Basic Questions of Dynamical Systems

1. What is the definition of a flow Φ_t on $\mathbb{R}^N$?

2. What is the definition of an evolution $\Psi_{t,s}$ on $\mathbb{R}^N$?

3. Which differential equation is associated to an evolution $\Psi_{t,s}$ and in what sense does the evolution “solve” the equation?

4. How can an evolution $\Psi_{t,s}$ on a vector space $\mathbb{R}^N$ be written as a flow? How can a flow Φ_t on a vector space $\mathbb{R}^N$ be written as an evolution?

5. How are α- and ω-limit sets of an initial value x_0 to a flow Φ_t defined?

6. What do we mean by equilibria, periodic orbits, heteroclinic orbits, and homoclinic orbits of a flow? What are their respective α- and ω-limit sets?

7. How is a first integral of a vector field defined? When do we call a first integral regular?

8. Which sufficient conditions ensure the existence of a regular first integral of a planar vector field?

9. What is an Euler multiplier of a vector field? How does it change orbits and solutions of the vector field?

10. How can the period of a periodic solution of the pendulum equation

$$\ddot{x} + V'(x) \;=\; 0,$$

$x \in \mathbb{R}$, be calculated?

11. What is an Hamiltonian vector field?

12. How do a flow Φ_t and the associated differential equation $\dot{x} = f(x)$ change under the coordinate transformation by a diffeomorphism $y = \Psi(x)$?

13. How can differential equations be solved by separation of variables? Formulate a theorem.

14. Formulate the parameter-dependent contraction-mapping theorem in a Banach space X. Include a statement about differentiability.

15. Formulate the theorem of Picard&Lindelöf on local existence and uniqueness of the solution to a differential equation.

16. How can the solution of the initial-value problem $\dot{x} = f(t,x)$, $x(t_0) = x_0$, be written as the fixed point of a map in a Banach space?

17. Let α, β, u be continuous, non-negative functions of $t \geq 0$, such that

$$u(t) \;\leq\; \alpha(t) + \int_0^t \beta(s)u(s)\,\mathrm{d}s,$$

for all $t \geq 0$. How can $u(t)$ be estimated in terms of α, β ?

18. Let I be the maximal interval of existence of the solution $x(t; x_0)$ of a trajectory to a locally Lipschitz vector field $f(x)$ with initial condition x_0. Is I open? Is I closed? Can I be bounded? What happens to the solution at the boundary of I ?

19. Which sufficient conditions ensure the existence of a global evolution to a non-autonomous vector field on $\mathbb{R}^n$?

20. Give an example of a vector field with a solutions that blows up in finite time.

21. Consider the solution $x(t; t_0, x_0)$ to the initial-value problem

$$\dot{x} \;=\; f(t,x), \qquad x(t_0) = x_0,$$

with smooth vector field f. Which differential equation is solved by the partial derivative

$$v(t; t_0, \xi) \;:=\; \frac{\partial}{\partial x_0} x(t; t_0, x_0)\xi$$

with respect to the initial condition x_0 ?

22. What is the Wronski-matrix evolution to a global solution $x(t)$ of a smooth vector field

$$\dot{x} \;=\; f(t,x)?$$

Which differential equation is solved by the Wronski matrices?

23. What is the explicit flow to the linear differential equation

$$\dot{x} \;=\; Ax,$$

with a constant matrix A in complex Jordan normal form?

24. How can the linear differential equation

$$x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x \;=\; 0,$$

with $x \in \mathbb{C}$ and constant complex coefficients a_j, be solved by an exponential ansatz?

25. What is the solution of the inhomogeneous linear system

$$\dot{x}(t) \;=\; Ax(t) + b(t), \qquad x(t_0) = x_0,$$

with constant matrix A and continuous function b ?

26. Consider an arbitrary C^1 vector field $\dot{x} = f(x)$ with $f(x_0) \neq 0$. What is the normal form of the flow in a local neighborbood of x_0 ?

27. Formulate and prove the flow-box theorem.

28. How are stability and asymptotic stability of an equilibrium defined?

29. How is the asymptotic stability of an equilibrium related to the linearization of the vector field?

30. In which sense is the flow near a hyperbolic equilibrium locally equivalent to its linearization? Formulate a theorem.

31. When are two linear systems

$$\dot{x} = Ax, \qquad \dot{y} = By,$$

$x, y \in \mathbb{R}^N$, called hyperbolic? When are they called C^0 flow equivalent? When are they called C^1 flow equivalent?

32. When do we call a set M positively invariant, negatively invariant, or invariant with respect to a given flow Φ_t?

33. Let the forward orbit $\gamma^+(x_0)$ of x_0 under a flow in $\mathbb{R}^N$ be bounded. Is the ω-limit set of x_0

 (a) open,
 (b) closed,
 (c) bounded,
 (d) compact,
 (e) stable,
 (f) asymptotically stable,
 (g) unstable?

 Why?

34. Let the forward orbit $\gamma^+(x_0)$ of x_0 under a flow in $\mathbb{R}^N$ be bounded. Is the ω-limit set of x_0

 (a) nonempty,
 (b) finite,
 (c) discrete,
 (d) connected,
 (e) positively invariant,
 (f) negatively invariant,
 (g) invariant?

 Why?

35. How is a Lyapunov function of a vector field defined? When do we call a Lyapunov function strict?

36. How does a Lyapunov function restrict possible ω-limit sets of a flow?

37. Formulate the invariance principle of LaSalle.

38. How are stability and asymptotic stability of a nonempty, compact, invariant set defined?

39. What is an attractor?

40. Formulate the theorem of Poincaré & Bendixson on ω-limit sets of bounded orbits of planar vector fields.

41. What is a limit cycle?

42. Formulate the criterion of Bendixson on the non-existence of periodic orbits of planar vector fields.

43. How many periodic orbits can a flow without equilibria on the plane possess?