Homework assignment

Dynamical Systems II

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 13: Let $\Phi : S^1 \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$ be a homeomorphism (in particular in general not C^2) with irrational rotation number $\varrho(\Phi) \notin \mathbb{Q}$. Let μ be an invariant Borel measure, i.e.

$$\mu(\Phi^{-1}(A)) \;=\; \mu(A), \qquad \text{for each Borel set } A \subseteq S^1.$$

Let μ be strictly positive on open subsets of S^1.

Prove: Φ is conjugated to a rigid rotation, i.e. there exists a homoemorphism $h : S^1 \to S^1$ such that

$$h\Phi h^{-1}(x) \;=\; x + 2\pi\varrho(\Phi) \pmod{2\pi}.$$

Problem 14: Consider the map $A : \mathbb{R} \to \mathbb{R}$,

$$A(y) \;=\; \begin{cases} 2y, & 0 \le y < 1 \\ A(y-1)+1, & 1 \le y \\ A(y+1)-1, & y < 0 \end{cases}$$

Thus $A(y+1) = A(y)+1$ for all y and A defines a map $\tilde{A} : S^1 \to S^1 = \mathbb{R}/\mathbb{Z}$. However A and $\tilde{A}$ are not homeomorphisms. Nonetheless, try to define the usual "rotation number" $\varrho(y_0)$ for initial conditions y_0. Does $\varrho(y_0)$ depend on y_0?

Problem 15: Find an example to show that a homeomorphism f of the circle with rational rotation number $\varrho(f)$ is not necessarily conjugated to a rigid rotation by the angle $\varrho(f)$.

Give an additional — necessary and sufficient — condition on f to be conjugated to a rigid rational rotation.

Problem 16: Consider the general pendulum equation

$$\ddot{x} + \nabla V(x) \;=\; 0, \qquad x \in \mathbb{R}^N,$$

with potential $V : \mathbb{R}^N \to \mathbb{R}$. Let $x = \dot{x} = 0$ be a *hyperbolic* equilibrium.

Consider the Hamiltonian $H(x, \dot{x}) = \frac{1}{2}\|\dot{x}\|^2 + V(x)$. Prove or disprove: Locally, the level set of the equilibrium is the union of its stable and unstable manifolds, i.e.

$$W^{\mathrm{u}}_{\mathrm{loc}}(0,0) \cup W^{\mathrm{s}}_{\mathrm{loc}}(0,0) \;=\; \{(x,\dot{x})\;;\; H(x,\dot{x}) = H(0,0)\}_{\mathrm{loc}}$$

(i) for one degree of freedom, $N = 1$;

(ii) for more degrees of freedom, $N > 1$.