Homework assignment

Dynamical Systems II

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 17: Consider the Fibonacci-Iteration

$$x_{n+1} = x_n + x_{n-1} \pmod 1$$

on the 2-torus $(x_n, x_{n-1}) \in (\mathbb{R}/\mathbb{Z})^2$, i.e., consider only the noninteger fractional part of the iteration. Is it well defined? Calculate stable and unstable manifolds of the fixed point $(0,0)$ under the iteration Ψ^n. Are they dense on the 2-torus?

Problem 18: Consider a diffeomorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ with $\Phi(0) = 0$. Let

$$\begin{aligned} W^s &= \{x \in \mathbb{R}^N \mid \lim_{n\to\infty} \Phi^n(x) = 0\} \\ W^u &= \{x \in \mathbb{R}^N \mid \lim_{n\to\infty} \Phi^{-n}(x) = 0\} \end{aligned}$$

denote the stable and the unstable set of the origin. Find — if possible — an example and a counterexample for each of the following cases:

(i) W^s is an embedded submanifold.

(ii) W^s is closed.

(iii) $W^s \cap W^u$ consists of exactly two distinct points.

Extra credit: Find an example such that W^s is not even a manifold.

Reminder: W is a manifold of dimension M if it is locally homeomorphic to $\mathbb{R}^M$. W is an *embedded submanifold* of dimension M in $\mathbb{R}^N$ if for all $x \in W$ there exists a ball $x \in B_\varepsilon(x) \subset \mathbb{R}^N$ and a homeomorphism $h : B_\varepsilon(x) \to h(B_\varepsilon(x)) \subset B_1(0)$ such that $h(W \cap B_\varepsilon(x)) = (0 \times \mathbb{R}^M) \cap h(B_\varepsilon(x))$. Here $B_1(0)$ is the unit ball in $\mathbb{R}^N$ centered at the origin.

Problem 19: Consider the pendulum

$$\ddot{\varphi} + \sin\varphi \; = \; 0.$$

Let

$$W^s_{loc} \; = \; \{(\varphi, \dot{\varphi}) = (\varphi, h(\varphi - \pi)) \; ; \; \pi - \varepsilon < \varphi < \pi + \varepsilon\}$$

be the local stable manifold at the equilibrium $\varphi = \pi$, $\dot{\varphi} = 0$. Determine the expansion

$$h(\psi) \; = \; \sum_{k=0}^{N} h_k \psi^k + \mathcal{O}(\psi^{N+1})$$

up to order $N = 3$.

Hint: Use the invariance of W^s.

Extra credit: Determine the corresponding expansion for the damped pendulum

$$\ddot{\varphi} + \alpha\dot{\varphi} + \sin\varphi \; = \; 0$$

with $\alpha > 0$.

Problem 20: Consider the damped pendulum

$$\ddot{\varphi} + \alpha\dot{\varphi} + \sin\varphi \; = \; 0$$

with $\alpha > 0$ and $\varphi \in \mathbb{R}$.

(i) Sketch the stable manifold of the equilibrium $\varphi = \pi$, $\dot{\varphi} = 0$ for $\alpha > 0$.

(ii) How do trajectories above and below the stable manifold differ?