Homework assignment

# Dynamical Systems II

Bernold Fiedler, Stefan Liebscher

http://dynamics.mi.fu-berlin.de/lectures/

**due date: Thursday, Dec 2, 2010**

**Problem 21:** Let X be a Banach space and $\Phi_n : X \to X$ be a $p$-periodic sequence of contractions, i.e. $\Phi_{n+p} = \Phi_n$ for all $n \in \mathbb{N}_0$.

(i) Prove that there exists a unique $x_0 \in X$, such that the sequence

$$x_{n+1} := \Phi_n(x_n), \qquad n \geq 0,$$

is periodic, i.e. $x_{n+p} = x_n$ for all $n \in \mathbb{N}_0$.

(ii) Take $\Phi_n = \Phi$ independent of $n \in \mathbb{N}_0$. Then any $x_n$ from (i) is a $p$-periodic point under iterations of $\Phi$; not only for $n = 0$, but also for any $0 \leq n < p$. But how is that possible, in view of the uniqueness result of (i)?

**Problem 22:** Let

$$\Sigma_N = \left\{ s = (s_j)_{j \in \mathbb{Z}} \mid s_j \in \{0, \ldots, N-1\} \right\}$$

be the set of sequences on $N$ symbols, with the metric

$$\mathrm{dist}\,(s, s') := \sum_{j \in \mathbb{Z}} (2N)^{-|j|} |s_j - s'_j|.$$

Consider the shift

$$\sigma : \Sigma_N \to \Sigma_N, \quad (s_j)_{j \in \mathbb{Z}} \mapsto (s_{j+1})_{j \in \mathbb{Z}}.$$

What are the fixed points of $\sigma$ ? Determine the stable and unstable sets and thus all homoclinic and heteroclinic orbits of these fixed points.

**Problem 23:** Calculate all fixed points of the bouncing-ball map $f$:

$$\begin{aligned} \Phi_{j+1} &= \Phi_j + v_j, \\ v_{j+1} &= \alpha\, v_j - \gamma \cos(\Phi_j + v_j), \end{aligned}$$

with $\Phi_j \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ and $v_j \in \mathbb{R}$, for $0 < \alpha < 1$ and $0 < \gamma$. How many fixed points does $f$ have for given $\alpha, \gamma$? Determine the type (stable, unstable, non-hyperbolic etc.) of the fixed points. Sketch the dependence of the fixed points on $\gamma$, for $\alpha = \frac{1}{2}$. What happens for $\alpha \to 1$?

**Problem 24:** Consider the space

$$\Sigma_2 \; = \; \left\{ s = (s_j)_{j\in\mathbb{Z}} \;\middle|\; s_j \in \{0,1\} \right\}$$

of sequences on the two symbols $\{0,1\}$, as described in class. The topology on $\Sigma_2$ is the product topology; it is generated by the cylinder sets

$$N_k(s) \; := \; \{ \; \tilde{s} \in \Sigma_2 \;\; | \;\; \tilde{s}_j = s_j \;\; \text{for all} \;\; |j| \le k \; \} \, , \qquad s \in \Sigma_2, \quad k \in \mathbb{N},$$

i.e. the open sets are all possible unions of arbitrary cylinder sets.

(i) Prove that the cylinder sets $N_k(s)$ are also closed. Thus there are *open and closed*.

(ii) Use this fact to prove that $\Sigma_2$ is totally disconnected, i.e. for arbitrary $s, \tilde{s} \in \Sigma_2$, $s \neq \tilde{s}$, there are open sets $U, \tilde{U} \subset \Sigma_2$, such that $s \in U$, $\tilde{s} \in \tilde{U}$, $U \cap \tilde{U} = \emptyset$, and $U \cup \tilde{U} = \Sigma_2$.