Homework assignment

# Dynamical Systems II

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http://dynamics.mi.fu-berlin.de/lectures/

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**Problem 25:** Consider the space

$$\Sigma_2 \;=\; \left\{ s = (s_j)_{j\in\mathbb{Z}} \;\middle|\; s_j \in \{0,1\} \right\}$$

of sequences on the two symbols $\{0,1\}$, as described in class. The topology on $\Sigma_2$ is the product topology; it is generated by the cylinder sets

$$N_k(s) \;:=\; \{\; \tilde{s} \in \Sigma_2 \;\;|\;\; \tilde{s}_j = s_j \;\text{ for all }\; |j| \le k \;\}\,, \qquad s \in \Sigma_2, \quad k \in \mathbb{N},$$

i.e. the open sets are all possible unions of arbitrary cylinder sets. For every sequence $c = (c_j)_{j\in\mathbb{Z}}$, $c_j > 0$, $\sum c_j < \infty$, define the metric $\mathrm{dist}_c$ on $\Sigma_2$ by

$$\mathrm{dist}_c\,(s, \tilde{s}) \;:=\; \sum_{j\in\mathbb{Z}} c_j \;|s_j - \tilde{s}_j|\,.$$

Choose one of the following claims and prove it:

(i) Given an arbitrary summable sequence $c$ of positive numbers. Then the metric $\mathrm{dist}_c$ induces the above defined topology on $\Sigma_2$.

(ii) Given two summable sequences $c, \tilde{c}$ of positive numbers. Then the metric $\mathrm{dist}_c$ is equivalent to $\mathrm{dist}_{\tilde{c}}$ if, and only if, there exists a positive constant $M > 0$ such that for all $j \in \mathbb{Z}$

$$\frac{1}{M} \;<\; \frac{\tilde{c}_j}{c_j} \;<\; M.$$

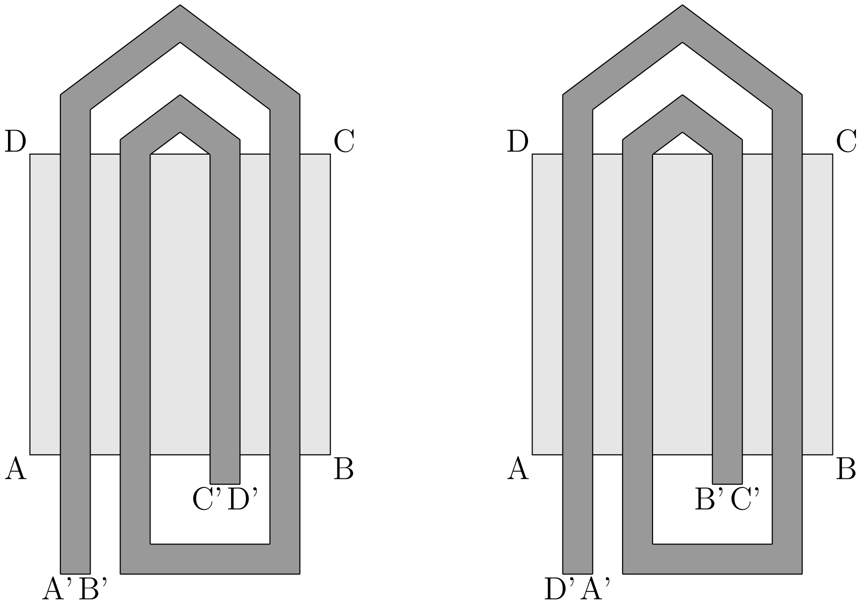
*Extra credit:* Prove the other claim.

Equivalent metrics induce the same topology, but metrics which induce the same topology need not be equivalent.

*Reminder:* The topology induced by a metric $\mu$ on a space $X$ is generated by all its $\varepsilon$-Balls, i.e. the open sets are all possible unions of arbitrary sets of the form

$$B_\varepsilon(x) \;:=\; \{\; \tilde{x} \in X \;\;|\;\; \mu(x, \tilde{x}) < \varepsilon \;\}\,, \qquad x \in X, \quad \varepsilon > 0.$$

**Problem 26:** Which of the following "paper-clip" maps gives rise to shift dynamics? (You can assume the maps to be affine linear, in the regions of intersection.)




**Problem 27:** Let the assumptions of the theorem about the $C^0$-horseshoe be satisfied for the iteration $\Phi$ on the square $Q$. Thus, there exists a homeomorphism $\tau$ conjugating the shift $\sigma : S \to S$ to $\Phi : I \to I$, on an invariant subset $I := \tau(S) \subset Q$. Let the horizontal and vertical Lipschitz-curves $U(s)$ and $V(s)$ be defined as in class, that is

$$\begin{aligned} U(s) &:= \left\{ q \in Q \;\middle|\; \Phi^{-k}(q) \in V_{s_k} \;\; \forall k \geq 1 \right\}, \\ V(s) &:= \left\{ q \in Q \;\middle|\; \Phi^{-k}(q) \in V_{s_k} \;\; \forall k \leq 0 \right\}, \end{aligned}$$

for any sequence $s = (s_k)_{k \in \mathbb{Z}} \in S$ and the primary vertical stripes $\{V_a \,|\, a \in A\}$ of the horseshoe construction.

Consider the unstable and stable sets of points $p \in I$,

$$\begin{aligned} W^{\mathrm{u}}(p) &:= \left\{ q \in Q \;\middle|\; \Phi^k(q) \in \bigcup_{a \in A} V_a \;\; \forall k \leq -1, \;\; \lim_{k \to -\infty} \operatorname{dist}\left(\Phi^k(p), \Phi^k(q)\right) = 0 \right\} \\ W^{\mathrm{s}}(p) &:= \left\{ q \in Q \;\middle|\; \Phi^k(q) \in \bigcup_{a \in A} V_a \;\; \forall k \geq 0, \;\; \lim_{k \to \infty} \operatorname{dist}\left(\Phi^k(p), \Phi^k(q)\right) = 0 \right\}. \end{aligned}$$

Let $p = \tau(s)$ be a point of the invariant set with corresponding sequence $s$. Prove:

(i) $U(s) \subset W^{\mathrm{u}}(p)$ and $V(s) \subset W^{\mathrm{s}}(p)$;

(ii) $U(s)$ and $V(s)$ are the connected components of $W^{\mathrm{u}}(p) \cap Q$ and $W^{\mathrm{s}}(p) \cap Q$ containing $p$. Thus, they are the local unstable and stable (Lipschitz) manifolds of $p$.

**Problem 28:** Let $\mu > 0$ be fixed. Consider a map $u : [0, 1] \to [0, 1]$ and the horizontal cone $S^+ = \{ (\xi, \eta) \ : \ |\eta| \leq \mu|\xi| \}$.

(i) Prove that $u$ is a horizontal curve (i.e. $u$ is Lipschitz continuous with Lipschitz constant $\mu$) if, and only if, its graph lies inside every horizontal cone attached to it (i.e. $\text{graph}\,(u) \subset S^+ + (x, u(x))$ for all $x \in [0, 1]$).

(ii) Let $u$ be differentiable. Prove that $u$ is a horizontal curve if, and only if, every tangent vector lies in $S^+$.

(iii) Formulate similar cone conditions for maps $v : \mathbb{R}^n \to \mathbb{R}^n$.