Homework assignment

# Dynamical Systems II

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**Problem 29:** Consider the Hénon map

$$\begin{aligned} x_{j+1} &= 1 - \alpha x_j^2 + \beta y_j, \\ y_{j+1} &= x_j. \end{aligned}$$

Find a horseshoe for $1 \ll \alpha$ and $0 < \beta \ll 1$.

*Hint:* $Q = [-0.1, 0.1] \times [-1, 1]$.

**Problem 30:** A measure of complexity of a map $\Phi$ is the *topological entropy* $h$: Let $N(n)$ be the number of periodic points of $\Phi$ with (not necessarily minimal) period $n$. Then the entropy is defined as

$$h := \limsup_{n\to\infty} \frac{\log N(n)}{n}.$$

Calculate the entropy $h$ of the shift on $m$ symbols. Prove that every iteration $\Phi$ containing a shift (i.e. with an invariant set $I$ such that $\Phi|_I$ is conjugate to a shift on $m$ symbols) has positive topological entropy.

**Problem 31:** Consider a diffeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ and a compact, invariant, hyperbolic set $I$. (In particular there exists a hyperbolic structure, i.e. continuous, invariant line bundles $L_p^{\pm}$, $p \in I$, with uniform rates of contraction/expansion.)

Prove or disprove: the hyperbolic structure on $I$ is unique.

**Problem 32:** [Horocycles] Consider the POINCARÉ-model of hyperbolic geometry, that is the upper half plane $\mathcal{H} = \{z = (x, y) \in \mathbb{R}^2 \,:\, y > 0\}$ with the arclength element

$$ds^2 \;=\; \frac{dx^2 + dy^2}{y^2}.$$

The geodesics (i.e. locally shortest paths) of $\mathcal{H}$ are the vertical straight lines, $\{z = (x, y) \,:\, x = c,\, y > 0\}$, $c \in \mathbb{R}$, and the (Euclidean) circles whith centers on the $x$-axis, $\{z = (x, y) \,:\, (x - c)^2 + y^2 = r,\, y > 0\}$, $c \in \mathbb{R}$, $r > 0$.

Consider the geodesic flow $\Phi$ on the unit tangent bundle $T^1\mathcal{H}$. Trajectories of $\Phi$ are given by geodesics of $\mathcal{H}$ with attached tangent unit vectors.

(i) Show that $(0, t)$ is a unit vector in $T_{(0,t)}\mathcal{H}$ and thus the curve $(z(t), \dot{z}(t)) = ((0, e^t), (0, e^t)) \in T^1\mathcal{H}$ is a trajectory of the geodesic flow.

(ii) Consider horizontal lines $W^s(t) := \{((x, e^t), (0, e^t)) \in T^1\mathcal{H}; x \in \mathbb{R}\}$. Show that horizontal lines are mapped onto horizontal lines under the geodesic flow. Show that horizontal lines are stable leaves (Blätter/fibers/manifolds), i.e. starting with $(w, \dot{w}) \in W^s(0)$ we have $\mathrm{dist}_{\mathcal{H}}(z(t), w(t)) \to 0$ as $t \to \infty$.

(iii) Prove that the inversion in the unit circle

$$\sigma(x, y) \;=\; \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right),$$

is an isometry of $\mathcal{H}$. Thus it maps trajectories of the geodesic flow onto trajectories. Use this to prove that the unstable leaves $W^u(t)$ of the trajectory (i) are given by circles. Sketch the families $W^u(t)$ and $W^s(t)$.

(iv) Horizontal translations

$$\tau_a(x, y) \;=\; (x + a, y)$$

are also isometries of $\mathcal{H}$. Choose $a \neq 0$ and sketch or plot the image of $z(t)$, $W^u(t)$, $W^s(t)$ under the isometry $\sigma \circ \tau_a$.