Homework assignment

Dynamical Systems II

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(As usual, solve (at least) 2 problems, get 1 right.)

Problem 49: Consider a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ of the flow φ_t with equilibrium $x_0 = 0$. Let $R : \mathbb{R}^n \to \mathbb{R}^n$ be a linear involution ($R \circ R = \mathrm{id}$). Assume that the vector field f is equivariant under the symmetry R, that is

$$f \circ R \;=\; R \circ f.$$

(i) Let the assumptions of the theorem on the existence of a *global* center manifold in $x_0 = 0$ be satisfied. Prove that the center manifold $W^c(x_0)$ inherits the symmetry of the vector field, that is

$$R(W^c) \;=\; W^c.$$

(ii) Let the assumptions of the theorem on the existence of a *local* center manifold in $x_0 = 0$ be satisfied. Prove that there exists a symmetric local center manifold $W^c_{\mathrm{loc}}(x_0)$, i.e.

$$R(W^c_{\mathrm{loc}}) \;=\; W^c_{\mathrm{loc}}.$$

Problem 50: Consider again the previous problem. Explain that the reduced vector field on the (symmetric) center manifold inherits the symmetry, i.e. it is equivariant under R restricted to the center eigenspace.

Then consider the vector field

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \;=\; f(x,y) \;=\; \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + g(x,y),$$

with $x \in \mathbb{R}$, $y \in \mathbb{R}^{N-1}$, an isolated equilibrium $(x,y) = (0,0)$, some hyperbolic matrix B, and C^1-terms $g(x,y) = \mathcal{O}(|x|^2 + \|y\|^2)$ of higher order. Assume that f is symmetric under the reflection at the y-hyperplane,

$$f \circ R \;=\; R \circ f \qquad \text{with} \quad R \;=\; \mathrm{diag}(-1, 1, \ldots, 1).$$

Show that in a local center manifold either $\omega(x,y) = 0$ for all nonzero (x,y) or $\alpha(x,y) = 0$ for all nonzero (x,y).

Problem 51: Discuss the “cusp bifurcation”

$$\dot{x} \;=\; x^3 + \lambda x + \mu, \qquad x, \lambda, \mu \in \mathbb{R}.$$

In particular, determine number and stability of equilibria for all parameters (λ, μ) and the parameter curves along which degenerate equilibria (i.e. saddle-node bifurcations) occur.

Problem 52: Consider the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\lambda_1 x \\ -\lambda_2 y \end{pmatrix}$$

with $0 \le \lambda_1 \le \lambda_2$. Determine all flow-invariant manifolds tangential to the eigenspace E_{λ_1} and their smoothness class $\mathcal{C}^k$ depending on λ_1, λ_2.

Relate your observations to the question of regularity of invariant manifolds correspon-ding to general eigenvalue splittings

$$\Re\text{e spec}\, A|_{E^s} \;\le\; -\beta_- \;<\; -\eta_- \;\le\; \Re\text{e spec}\, A|_{E^c} \;\le\; \eta_+ \;<\; \beta_+ \;\le\; \Re\text{e spec}\, A|_{E^u}$$

of the linearization A at an equilibrium.

Problem 53: Consider the system

$$\begin{aligned} \dot{x}_\mathrm{c} &= Ax_\mathrm{c} + f(x_\mathrm{c} + x_\mathrm{h}), \\ \dot{x}_\mathrm{h} &= Bx_\mathrm{h} + g(x_\mathrm{c} + x_\mathrm{h}), \end{aligned}$$

with $f, g \in C^\kappa$, $f(x) = \mathcal{O}(|x|^2)$, $g(x) = \mathcal{O}(|x|^2)$, and $\mathrm{spec}(A) \subset \mathbf{i}\mathbb{R}$, $\mathrm{spec}(B) \cap \mathbf{i}\mathbb{R} = \emptyset$. Assume the existence of a local center manifold $x_\mathrm{h} = h(x_\mathrm{c})$, $h \in C^\kappa$.

Prove that the κ-th derivative of h is uniquely determined. Describe a method to calculate the Taylor expansion of h.

Hint: Compare the Taylor expansions of $Bh(x_\mathrm{c}) + g(x_\mathrm{c} + h(x_\mathrm{c}))$ and $Dh(x_\mathrm{c})\,[Ax_\mathrm{c} + f(x_\mathrm{c} + h(x_\mathrm{c}))]$ and use the fact that $h(x_\mathrm{c}) = \mathcal{O}(|x_\mathrm{c}|^2)$.