Voluntary problems

Dynamical Systems II

Bernold Fiedler, Stefan Liebscher
http://dynamics.mi.fu-berlin.de/lectures/

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Problem X1: Let $\Phi : M \to M$ be a continuous map on a metric space M. We call a sequence $(\xi_k)_{k\in\mathbb{N}}$ a δ-*pseudo orbit*, if the estimate

$$\mathrm{dist}(\Phi(\xi_k), \xi_{k+1}) \ < \ \delta.$$

holds for all $k \in \mathbb{N}$. We call a Φ-orbit $(x_k)_{k\in\mathbb{N}} = (\Phi^k(x_0))_{k\in\mathbb{N}}$ in M an ε-*shadow* of the pseudo orbit $(\xi_k)_{k\in\mathbb{N}}$ if the estimate

$$\mathrm{dist}(x_k, \xi_k) \ < \ \varepsilon$$

holds for all $k \in \mathbb{N}$. We say that the pair (M, Φ) has the *shadow property*, if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that every δ-pseudo orbit has an ε-shadow.

(i) Prove: the shift on two symbols has the shadow property.

(ii) Give an interpretation of the shadow property from a numerical view point.

Extra credit: Is the shadow unique?

Problem X2: Consider the bouncing-ball map f,

$$\begin{aligned} \Phi_{j+1} &= \Phi_j + v_j, \\ v_{j+1} &= \alpha\, v_j - \gamma \cos(\Phi_j + v_j), \end{aligned}$$

discussed in class, with $\alpha = 1$ and $\gamma > 0$. Choose γ large enough and find a "horseshoe" for $(\Phi, v) \in (0, 4\pi) \times \mathbb{R}$ giving rise to an invariant set I such that $f|I$ is conjugate to the shift on 4 symbols.

Extra credit: Find a shift on m symbols for every $m \geq 2$.

Problem X3: Consider again the bouncing-ball map f:

$$\begin{aligned} \Phi_{j+1} &= \Phi_j + v_j, \\ v_{j+1} &= \alpha\, v_j - \gamma \cos(\Phi_j + v_j) \end{aligned}$$

with $(\Phi, v) \in S^1 \times \mathbb{R} = (\mathbb{R}/(2\pi\mathbb{Z})) \times \mathbb{R}$.

What is the (non-unique!) lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ of f? Choose a suitable lift and modify the horseshoe construction done in class (or in the solution to the previous problem) such that $v > 0$ on the induced invariant set.

Extra credit: Give a physical interpretation.

Problem X4: Consider again the bouncing-ball map $f_{\alpha,\gamma}$ on $S^1 \times \mathbb{R}$:

$$\begin{aligned} \Phi_{j+1} &= \Phi_j + v_j, \\ v_{j+1} &= \alpha\, v_j - \gamma \cos(\Phi_j + v_j), \end{aligned}$$

with $0 < \alpha < 1$ and $0 < \gamma$. Define the domain

$$D := \left\{ (\Phi, v) \in S^1 \times \mathbb{R} \; : \; |v| \leq \frac{\gamma}{1-\alpha} + \varepsilon \right\}$$

for some $\varepsilon > 0$. Prove:

(i) D is positively invariant, i.e. $f_{\alpha,\gamma}(D) \subseteq D$.

(ii) D is absorbing, i.e. for all (Φ_0, v_0) there exists $n_0 \in \mathbb{N}$ such that $(\Phi_n, v_n) \in D$ for all $n \geq n_0$.

(iii) The *global attractor*, defined by

$$\mathcal{A}_{\alpha,\gamma} := \bigcap_{n=0}^{\infty} f^n_{\alpha,\gamma}(D),$$

is compact and invariant under $f_{\alpha,\gamma}$ as well as $f^{-1}_{\alpha,\gamma}$. Furthermore, $\mathcal{A}_{\alpha,\gamma}$ is the *maximal* compact and invariant set.

(iv) $\mathcal{A}_{\alpha,\gamma}$ is indeed attracting, i.e. for all (Φ_0, v_0)

$$\lim_{n\to\infty} \operatorname{dist}\Big((\Phi_n, v_n), \mathcal{A}_{\alpha,\gamma} \Big) = 0.$$

(v) $\mathcal{A}_{\alpha,\gamma}$ contains the closure of the set of all periodic points of $f_{\alpha,\gamma}$.

Problem X5: Consider the iteration on the 2-torus $T = (\mathbb{R}/\mathbb{Z})^2$ defined by the matrix

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Find a horseshoe for a suitable iterate B^k, $k > 0$.

Hint: Identify the torus with the unit square centered at $(0,0)$ and investigate the images of a parallelogram parallel to the eigenvectors of B.

Problem X6: [(infinite) adding machine] Consider the space

$$\Sigma_2^+ = \left\{ s = (s_j)_{j \in \mathbb{N} \cup \{0\}} \;\middle|\; s_j \in \{0,1\} \right\}$$

of (one sided) sequences on the two symbols $\{0,1\}$. The topology on Σ_2^+ is the product topology, as for the space of two-sided sequences defined in class. It is generated by the cylinder sets

$$N_k(s) := \left\{ \tilde{s} \in \Sigma_2^+ \;\middle|\; \tilde{s}_j = s_j \;\; \forall j \le k \right\}, \qquad s \in \Sigma_2^+, \quad k \in \mathbb{N},$$

and an equivalent metric would be

$$\operatorname{dist}(s, \tilde{s}) := \sum_{j \ge 0} 2^{-j} \, |s_j - \tilde{s}_j| \, .$$

For an arbitrary but fixed element $g \in \Sigma_2^+$ define the (addition) map

$$T_g : \Sigma_2^+ \longrightarrow \Sigma_2^+, \qquad a = (a_n)_{n \in \mathbb{N} \cup \{0\}} \longmapsto T_g(a) = (T_g(a)_n)_{n \in \mathbb{N} \cup \{0\}}$$

recursively by

$$\begin{aligned} r_0 &:= 0, \\ T_g(a)_n + 2 r_{n+1} &:= a_n + g_n + r_n, \qquad n = 0, 1, \ldots \end{aligned}$$

Here the temporary variable $r \in \Sigma_2^+$ denotes the overflow of the addition. If one takes *finite* sequences as binary representations of integer numbers then $T_g(a)$ is just the sum of a and g.

(i) Let $g_0 = 0$. Prove that Σ_2^+ does not contain a dense orbit of T_g. (Find, for example, nontrivial invariant subsets of Σ_2^+.)

(ii) Let $g = (1, 0, 0, 0, \ldots)$. Find a dense orbit.

Extra credit: Prove that T_g has a dense orbit for arbitrary g with $g_0 = 1$. In this case, *each* orbit of T_g is dense in Σ_2^+. However there are no periodic orbits and no sensitive dependence on initial conditions, thus no "chaos".

Problem X7: [Subshift of finite type] Let $\Sigma_N = \{0, 1, \ldots, N-1\}^{\mathbb{Z}}$ be the set of 2-sided sequences on N symbols and $\sigma : \Sigma_N \to \Sigma_N$, $(x_k)_{k\in\mathbb{Z}} \mapsto (x_{k+1})_{k\in\mathbb{Z}}$ the shift.

Consider a matrix $A = (a_{k\ell})_{0\le k,\ell\le N-1} \in \{0,1\}^{N\times N}$ with entries 0 or 1. Let every row and every column of A contain at least one nonzero entry. Define the set

$$\Sigma_A := \left\{ x = (x_k)_{k\in\mathbb{Z}} \in \Sigma_N \;\middle|\; a_{x_k x_{k+1}} = 1 \text{ for all } k \in \mathbb{Z} \right\}$$

of sequences respecting the transfer matrix A. (The transfer matrix A determines valid successors x_{k+1} of elements x_k in sequences $x \in \Sigma_A$.)

Note that Σ_A is nonempty and invariant under the shift σ. The shift σ on the set Σ_A is also called subshift of finite type.

(i) Let

$$\Sigma_{A,n,\alpha,\beta} := \{ (x_0, x_1, \ldots, x_n) \mid x \in \Sigma_A,\ x_0 = \alpha,\ x_n = \beta \}$$

be the set of finite sequences of length $n+1$ which start with α and end with β. Prove that the number of elements of $\Sigma_{A,n,\alpha,\beta}$ is given by the corresponding entry of A^n, i.e.

$$|\Sigma_{A,n,\alpha,\beta}| = (A^n)_{\alpha\beta}.$$

(ii) Prove that the topological entropy if determined by the largest eigenvalue of A, i.e.

$$h_{\text{top}} = \log|\lambda_{\max}(A)|.$$

Reminder: We defined the *topological entropy* h in Problem 30 as follows: Let $N(n)$ be the number of periodic points of σ with (not necessarily minimal) period n. Then the entropy is defined as

$$h_{\text{top}} := \limsup_{n\to\infty} \frac{\log N(n)}{n}.$$

Problem X8: Consider again the subshift of finite type, i.e. the shift σ on the set of sequences

$$\Sigma_A := \left\{ x = (x_k)_{k\in\mathbb{Z}} \in \Sigma_N \;\middle|\; a_{x_k x_{k+1}} = 1 \text{ for all } k \in \mathbb{Z} \right\}$$

for a given transfer matrix $A = (a_{k\ell})_{0\le k,\ell\le N-1} \in \{0,1\}^{N\times N}$ with entries 0 or 1. Let every row and every column of A contain at least one nonzero entry.

Prove that σ possesses a dense orbit in Σ_A if, and only if, for every $0 \le k, \ell \le N-1$ there exists a positive integer n, such that the matrix entry $(A^n)_{k,\ell}$ is nonzero.