Homework assignment

Dynamical Systems III

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http://dynamics.mi.fu-berlin.de/lectures/

due date: Friday, April 22, 2011, 13:00

(As usual, solve (at least) 2 problems, get 1 right.)

Aufgabe 1: Let $H_2(\mathbb{R}^2)$ be the space of monomials of degree 2 in two variables x, y, that is

$$H_2(\mathbb{R}^2) \;=\; \operatorname{span}\left\{ \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix} \right\}.$$

Determine the image of $\operatorname{ad} A(H_2(\mathbb{R}^2))$ with

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Aufgabe 2: Consider the Lie algebra $\mathfrak{so}(3,\mathbb{R})$ of anti-symmetric real (3×3) matrices. Show that the map

$$\mathfrak{so}(3,\mathbb{R}) \longrightarrow \mathbb{R}^3, \qquad \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & c & 0 \end{pmatrix} \longmapsto \begin{pmatrix} -c \\ b \\ -a \end{pmatrix}$$

transforms the Lie bracket

$$[\mathfrak{a}, \mathfrak{b}] = \mathfrak{a}\mathfrak{b} - \mathfrak{b}\mathfrak{a}$$

into the vector product on $\mathbb{R}^3$. Give a geometric interpretation.

Aufgabe 3: Prove that for every $\mathfrak{a} \in \mathfrak{so}(3,\mathbb{R})$

$$A(t) = \exp(\mathfrak{a}t)$$

defines a one-parameter subgroup $\{A(t), t \in \mathbb{R}\}$ of $SO(3,\mathbb{R})$, the Lie group of orthogonal real (3×3) matrices with determinant 1.

Remark: The tangent space of $SO(3,\mathbb{R})$ at the unit matrix Id is given by $\mathfrak{so}(3,\mathbb{R})$. Every fixed element $\mathfrak{a} \in \mathfrak{so}(3,\mathbb{R})$ of the Lie algebra defines a particular vector field on the Lie group, by applying the Lie group. $A(t)$ is the integral courve of this vector field with initial value Id. Contrary to the group of diffeomorphisms, exp is locally surjective in this case.

Aufgabe 4: Let A be a real matrix and

$$\Gamma = \operatorname{clos}\{\, \exp(At) \mid t \in \mathbb{R} \,\}$$

be compact. Prove that Γ is a torus, i.e. $\Gamma = \mathbb{R}^k/\mathbb{Z}^k$ for an appropriate k.