Homework assignment

Dynamical Systems III

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http://dynamics.mi.fu-berlin.de/lectures/

due date: Friday, April 29, 2011, 13:00

(As usual, solve (at least) 2 problems, get 1 right.)

Aufgabe 5: Consider a vector field

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix},$$

with $f_1, f_2 = \mathcal{O}(|x_1|^2 + |x_2|^2)$.

(i) Calculate a normal form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(x_1, x_2) \end{pmatrix} + \text{h.o.t.},$$

up to order 2, either by choosing a suitable complement to $\mathrm{ad}(A)$ or by subsequent coordinate transformation. This truncated normal form can be written as a second-order equation, $\ddot{x}_1 = g(x_1, \dot{x}_1)$.

(ii) A two-parameter unfolding of the bifurcation is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}.$$

The resulting normal form (as given by (i) with the additional parameters) can be scaled by

$$\begin{aligned} x_1 &= \sigma^2 \tilde{x}_1, \\ x_2 &= \sigma^3 \tilde{x}_2, \\ \lambda &= \sigma^4 \tilde{\lambda}, \\ \mu &= \sigma^2 \tilde{\mu}, \\ t &= \sigma^{-1} \tilde{t}. \end{aligned}$$

Determine the rescaled normal form in $(\tilde{x}_1, \tilde{x}_2, \tilde{\lambda}, \tilde{\mu}, \tilde{t})$. up to order one in σ. Sketch the phase portrait of the Hamiltonian(!) vector field given by the rescaled normal form to order zero in σ for relevant parameter values. Determine curves of saddle-node and Poincaré-Andronov-Hopf bifurcations in the (λ, μ) parameter plane.

Extra credit: Are the Hopf points subcritical or supercritical? How do the perturbation terms of order one in σ change the phase portraits?

Aufgabe 6: The group $GL(n, \mathbb{R})$ acts on the space $M^{n\times n}(\mathbb{R})$ of $(n \times n)$-matrices by conjugation:

$$\forall G \in GL(n),\ A \in M^{n\times n} \quad : \quad T_G A \ :=\ G^{-1}AG.$$

The conjugacy classes $\{T_G A \,|\, G \in GL(n)\}$, $A \in M^{n\times n}$, thus become orbits of the group action. A normal form chooses a section to these group orbits.

Compare the Jordan normal form, $n = 2$, with the section given by the 2-parameter family

$$\begin{pmatrix} 0 & 1 \\ a_1 & a_2 \end{pmatrix}, \qquad (a_1, a_2) \in \mathbb{R}^2$$

(Arnold normal form). Discuss continuity, differentiability, and smoothness.

Aufgabe 7: Define the scalar product of $(n \times n)$-matrices $A, B \in M^{n\times n}(\mathbb{C})$:

$$\langle A, B\rangle \ :=\ \text{trace}\,(AB^*),$$

with the conjugate transpose (or adjoint matrix) $B^* = \overline{B^{\mathrm{T}}}$. Define the action

$$\text{ad}(A)\,B \ :=\ [A, B] \ =\ AB - BA.$$

(i) Prove: $\text{ad}(A)^* = \text{ad}(A^*)$, i.e.

$$\forall B, C \in M^{n\times n}(\mathbb{C}) \ :\ \langle \text{ad}(A)\,B,\ C\rangle = \langle B,\ \text{ad}(A^*)\,C\rangle.$$

(ii) Calculate the kernel of ad $(A)^*$ for

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \text{with} \quad 0 \neq \lambda \neq \mu \neq 0.$$

(iii) Prove: the map

$$\Psi_A \ :\ GL(n) \longrightarrow M^{n\times n}(\mathbb{C}), \qquad G \longmapsto G^{-1}AG$$

is analytic with linearization

$$D\Psi_A(\text{id}) \ :\ M^{n\times n}(\mathbb{C}) \longrightarrow M^{n\times n}(\mathbb{C}), \qquad C \longmapsto \text{ad}(A)\,C.$$

Aufgabe 8: Consider a vector field

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + f(x_1, x_2, x_3),$$

with $f = \mathcal{O}(\|x\|^2)$. Determine the remaining monomials of the normal form introduced in class. Unfold the linearization with two parameters. Determine curves of saddle-node and Poincaré-Andronov-Hopf bifurcations in the parameter plane.

Extra credit: Sketch the phase portrait of the truncated normal form for relevant parameter values.