Homework assignment

Dynamical Systems III

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http://dynamics.mi.fu-berlin.de/lectures/

due date: Friday, May 6, 2011, 13:00

(As usual, solve (at least) 2 problems, get 1 right.)

Aufgabe 9: Consider the logistic map

$$F : [0,1] \to [0,1], \qquad F(x) = ax(1-x),$$

for parameter values $1 \leq a \leq 4$. Let $x^*(a)$ be the branch of nontrivial fixed points that bifurcates transcritically from the trivial equilibrium $x = 0$, in $a = 1$. Let $a^* > 1$ the parameter value of the first bifurcation along the branch $x^*(a)$.

(i) Transform to coordinates $y = x - x^*(a)$, $\lambda = a - a^*$.

(ii) Determine the normal form of (i) in $(y, \lambda) = (0, 0)$, at least to order 3.

(iii) Do the solutions of double period bifurcate subcritically or supercritically?

Aufgabe 10: Prove that the exponential map

$$\exp : \mathfrak{sl}(2, \mathbb{R}) \to SL(2, \mathbb{R})$$

is ***not*** surjective but locally (near *id*) surjective.

Remark: $SL(2, \mathbb{R})$ denotes the group of real (2×2)-matrices with determinant 1 and $\mathfrak{sl}(2, \mathbb{R})$ denotes the algebra of trace-free (2×2)-matrices.

Aufgabe 11: Consider a parameter dependent vector field,

$$(1) \qquad \begin{array}{rcll} \dot{x} &=& f(x, \lambda), & x \in \mathbb{R}^n, \\ \dot{\lambda} &=& 0, & \lambda \in \mathbb{R}^m, \end{array}$$

in the extended phase space $\mathbb{R}^{n+m}$. Assume a trivial equilibrium, $f(0, \lambda) \equiv 0$. Thus, the linearization at the origin yields

$$A = \left(\begin{array}{c|c} D_x f(0,0) & 0 \\ \hline 0 & 0 \end{array} \right).$$

Compare the $\mathrm{ad}(A^{\mathrm{T}})$ normal form of the full system with the $\mathrm{ad}(A^{\mathrm{T}})$ normal form of

$$(2) \qquad \dot{x} = f(x, 0).$$

Prove that replacing the coefficients of the normal form to (2) by suitable polynomials in λ yields the normal form to (1).

Aufgabe 12: Consider a vector field

$$\dot{x} = f(x), \qquad f(0) = 0,$$

with diagonalizable linearization $A = Df(0)$ at the equilibrium $x = 0$. Let $x = (x_s, x_u, x_c)$ be the splitting according to the spectral decomposition of A. Let

$$\begin{pmatrix} \dot{y}_s \\ \dot{y}_u \\ \dot{y}_c \end{pmatrix} = \begin{pmatrix} A_s & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_c \end{pmatrix} \begin{pmatrix} y_s \\ y_u \\ y_c \end{pmatrix} + \begin{pmatrix} g_s(y_s, y_u, y_c) \\ g_u(y_s, y_u, y_c) \\ g_c(y_s, y_u, y_c) \end{pmatrix} + \mathcal{O}(\|y\|^{N+1})$$

be the $\mathrm{ad}(A^{\mathrm{T}})$ normal form to order $N \geq 2$.

Prove that $g_s(0, 0, y_c) \equiv 0$ and $g_u(0, 0, y_c) \equiv 0$.

Hint: Discuss the kernel of $\mathrm{ad}(A^{\mathrm{T}})$. Show that pure monomials in y_c cannot appear in the stable/unstable components.

Remark: Thus, in normal form, the center eigenspace is invariant and yields a center manifold. The normal form of the reduced vector field is given by $\dot{y}_c = A_c y_c + g_c(0, 0, y_c)$.