Homework assignment

Dynamical Systems III

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http://dynamics.mi.fu-berlin.de/lectures/

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(As usual, solve (at least) 2 problems, get 1 right.)

Consider a Lie group $\mathfrak{G}$ with unit element E and its Lie algebra $\mathfrak{g} = T_E\mathfrak{G}$, as defined in class. Any group element $H \in \mathfrak{G}$ acts on $\mathfrak{G}$ by composition and conjugation, i.e.

$$\begin{array}{rcll} \circ H & : & \mathfrak{G} \to \mathfrak{G}, & G \mapsto G \circ H, \\ H\circ & : & \mathfrak{G} \to \mathfrak{G}, & G \mapsto H \circ G, \\ H^* & : & \mathfrak{G} \to \mathfrak{G}, & G \mapsto H^{-1} \circ G \circ H, \end{array}$$

where H^{-1} denotes the inverse of H in $\mathfrak{G}$. Linearization yields actions on the tangent bundle, i.e.

$$\begin{array}{rcll} \circ H & : & T_G\mathfrak{G} \to T_{G\circ H}\mathfrak{G}, & g \mapsto g \circ H, \\ H\circ & : & T_G\mathfrak{G} \to T_{H\circ G}\mathfrak{G}, & g \mapsto H' \circ g, \\ H^* & : & \mathfrak{g} \to \mathfrak{g}, & g \mapsto (H^{-1})' \circ g \circ H. \end{array}$$

Finally, the exponential map

$$\exp : \mathfrak{g} \times \mathbb{R} \to \mathfrak{G}, \qquad (g, t) \mapsto \exp(g \cdot t)$$

is given by the solution of

$$\tfrac{\mathrm{d}}{\mathrm{d}t} G(t) \;=\; g \circ G(t), \qquad G(0) \;=\; E.$$

Problem 13: Prove that for each Lie-algebra element $g \in \mathfrak{g}$ and real numbers s, t

$$\exp(g \cdot (st)) \;=\; \exp((sg) \cdot t) \;=\; \exp((stg) \cdot 1)$$

Problem 14: We define the adjoint action of $H \in \mathfrak{G}$ on $\mathfrak{g}$ as the inverse of H^*

$$Ad(H) \;:\; \mathfrak{g} \to \mathfrak{g}, \qquad g \mapsto H' \circ g \circ H^{-1}.$$

Prove that $Ad : \mathfrak{G} \to GL(\mathfrak{g})$ is a group homomorphism, i.e. $Ad(H)$ is linear and invertible, and $Ad(H_1) \circ Ad(H_2) = Ad(H_1 \circ H_2)$.

Problem 15: Assume that $\mathfrak{G}$ is a matrix group, and consider the adjoint action $ad : \mathfrak{g} \to gl(\mathfrak{g}) = End(\mathfrak{g})$,

$$ad(h) \quad : \quad \mathfrak{g} \to \mathfrak{g}, \qquad g \mapsto \tfrac{\mathrm{d}}{\mathrm{d}t} Ad(\exp(h \cdot t))\big|_{t=0}\, g.$$

Prove that $ad(h)g = [h, g] = hg - gh$, and ad is a homomorphism of Lie algebras.

Extra creadit: for an arbitrary Lie algebra $\mathfrak{g}$ define the adjoint action by the Lie bracket, i.e. $ad(h)g := [h, g]$. Then $ad : \mathfrak{g} \to gl(\mathfrak{g})$ is a homomorphism of Lie algebras.

Problem 16: Consider the Lie algebra of Taylor jets of vector fields up to order $m > 1$,

$$\begin{aligned} \mathfrak{g} &:= \{T_m f \mid f(x) = Ax + \cdots,\ f \in \mathcal{C}^\infty(A),\ A \in Mat(N, \mathbb{R})\} \\ &= Mat(N, \mathbb{R}) \times H_2(\mathbb{R}^N) \times \cdots \times H_n(\mathbb{R}^N). \end{aligned}$$

Identify a Talor jet $\tilde{f} = T_m f \in \mathfrak{g}$ with the vector of its coefficients, ordered with respect to the degree of their monomials. Then, any linear map on $\mathfrak{g}$ can be represented as a matrix.

Prove that the adjoint action $ad(f)$ has the form of a block-triangular matrix with blocks correspondig to the spaces $H_k(\mathbb{R})$ of homogeneous polynomials,

$$ad(f) \;=\; \begin{pmatrix} ad_1 A & 0 & & \cdots & 0 \\ * & ad_2 A & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ * & \cdots & * & ad_{n-1} A & 0 \\ * & \cdots & & * & ad_n A \end{pmatrix}$$