Homework assignment

Dynamical Systems III

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http://dynamics.mi.fu-berlin.de/lectures/

due date: Friday, May 20, 2011, 13:00

(As usual, solve (at least) 2 problems, get 1 right.)

Problem 17: What does the Theorem of Crandall & Rabinowitz tell about the system

$$\dot{x} = f(\lambda, x), \qquad \lambda \in \mathbb{R},\ x \in \mathbb{R}^N, \qquad f(\lambda, 0) = 0, \qquad f \in \mathcal{C}^2,$$

provided the linearization $A(\lambda) = \mathrm{D}_x f(\lambda, 0)$ in $\lambda = 0$ has a geometrically simple but algebraically double eigenvalue 0.

Extra credit: Have we thus understood the Bogdanov-Takens bifurcation?

Problem 18: Let X be a Banach space and $X_1, X_2 \subseteq X$.

(i) Prove the equivalence of the following statements:

(a) The exists a bounded linear projection $P : X \to X$, i.e. $P^2 = P$, with $X_1 = \mathrm{Ker}(P)$ and $X_2 = \mathrm{Im}(P)$.

(b) X_1, X_2 are closed, linear subspaces of X, with $X = X_1 \oplus X_2$.

(ii) Assume (i)(a). Let $Q = \mathrm{id} - P$. Prove or disprove:

(a) $P^3 = P$
(b) $P^3 = Q$
(c) $P^3 = \mathrm{id}$
(d) $Q^2 = P$
(e) $Q^2 = Q$
(f) $Q^2 = \mathrm{id}$
(g) $[P, Q] = P$
(h) $[P, Q] = Q$
(i) $[P, Q] = \mathrm{id}$
(j) $(\mathrm{id} - 2P)^2 = P$
(k) $(\mathrm{id} - 2P)^2 = Q$
(l) $(\mathrm{id} - 2P)^2 = \mathrm{id}$
(m) $\sum_{k=0}^{n-1} \binom{n}{k} P^k Q^{n-k} = Q$
(n) $\sum_{k=0}^{n-1} 2^k \binom{n}{k} P^k Q^{n-k} = P$

Problem 19: Determine all Banach spaces X, Y that possess a bounded linear operator $K : X \to Y$ which is, both, compact and Fredholm. Determine the Fredholm index of every such K.

Problem 20: Consider the inverse of the derivative,

$$\left(\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} f\right)(x) := \int_0^x f(\xi)\mathrm{d}\xi,$$

as a map

(i) $\mathcal{BC}^0([0,1]) \to \mathcal{BC}^0([0,1])$,
(ii) $\mathcal{BC}^0([0,1]) \to \mathcal{BC}^1([0,1])$,
(iii) $\mathcal{BC}^0([0,1]) \to \mathcal{BC}^1_0([0,1]) := \{f \in \mathcal{BC}^1([0,1]) \,|\, f(0) = 0\}$.

Which of these maps is quasi-Fredholm? Which of these maps is Fredholm? What is the Fredholm index?