Homework assignment

Dynamical Systems III

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http://dynamics.mi.fu-berlin.de/lectures/

due date: Friday, May 27, 2011, 13:00

(As usual, solve (at least) 2 problems, get 1 right.)

Problem 21: We want to solve

$$x(t) = \int_0^t k(t,s)x(s)\mathrm{d}s + f(t), \qquad x(0) = 0,$$

for given continuous functions

$$k \in \mathcal{C}([0,1]^2, \mathbb{R}) \quad \text{and} \quad f \in \mathcal{C}_0([0,1],\mathbb{R}) := \mathcal{C}([0,1],\mathbb{R}) \cap \{\chi | \chi(0) = 0\}.$$

Prove existence and uniqueness of a solution $x \in \mathcal{C}_0([0,1],\mathbb{R})$.

Hint: Write the problem as $\mathcal{L}x = f$, with a linear operator $\mathcal{L}$. Show that $\mathcal{L}$ is Fredholm with index zero. Show that $\mathcal{L}$ has trivial kernel.

Problem 22: Consider a map

$$f : \mathbb{R} \times X \to Y, \qquad (\lambda, x) \mapsto f(\lambda, x), \qquad f(\lambda, 0) = 0.$$

Let the assumptions of the theorem of Crandall & Rabinowitz hold. Discuss the bifurcation equation after Lyapunov-Schmidt reduction: How can a transcritical bifurcation be distinguished from a pitchfork bifurcation? Folmulate a suitable non-degeneracy condition in the theorem of Crandall & Rabinowitz to ensure a transcritical bifurcation, i.e. a bifurcating family of nontrivial zeros of the form $(\lambda(s), x(s)) = (s, x(s))$.

Extra credit: How can subcritical and supercritical pitchfork bifurcations be distinguished, if above non-degeneracy condition is violated?

Problem 23: Let $k \in \mathbb{Z}$ and

$$L : \mathcal{C}^2([0,\pi]) \cap \{0 = x'(0) = x'(\pi)\} \to \mathcal{C}^0([0,\pi]), \qquad x \mapsto x'' + k^2 x.$$

We know that L is Fredholm with index 0. Prove that

$$\text{Range } L \;=\; \langle \sigma \mapsto \cos k\sigma \rangle^{\perp}.$$

Problem 24: Consider again the Euler rod discussed in class,

$$f(\lambda, x)(\sigma) \;=\; x''(\sigma) + \lambda \sin(x(\sigma)) \;\equiv\; 0,$$

where f is considered as a map

$$f \,:\, \mathbb{R} \times \mathcal{C}^2([0,1]) \cap \{0 = x'(0) = x'(1)\} \to \mathcal{C}^0([0,1]).$$

(i) Discuss the bifurcation at $\lambda = 0$.

(ii) Consider f on the space

$$\mathcal{C}^2([0,1]) \cap \{0 = x'(0) = x'(1)\} \cap \left\{ \int_0^1 x(\sigma)\mathrm{d}\sigma = 0 \right\}.$$

Give a physical interpretation of the additional constraint. Discuss solutions and bifurcations of the new problem. In particular, find a suitable image space to apply the theorem of Crandall & Rabinowitz.