Homework assignment

Dynamical Systems III

Bernold Fiedler, Stefan Liebscher

http://dynamics.mi.fu-berlin.de/lectures/

due date: Friday, June 03, 2011, 13:00

(As usual, solve (at least) 2 problems, get 1 right.)

Problem 25: Consider the space $L_2(S^1, \mathbb{R})$, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, of periodic, square-integrable, real-valued funtions. The canonical representation of the group $SO(2) = S^1$ on $L_2(S^1)$ is given by

$$(\gamma f)(x) = f(\gamma + x).$$

Determine all invariant subspaces of $L_2(S^1, \mathbb{R})$ on which $SO(2)$ acts irreducibly.

Hint & extra credit: Consider the Fourier transform $T : L_2(S^1, \mathbb{R}) \to \ell_2(\mathbb{C})$. Determine the representation of $SO(2)$ on ℓ_2 induced by T.

Problem 26: Consider the group Γ of automorphisms of the cube $[-1, 1]^3$ in $\mathbb{R}^3$. Choose either orientation preserving automorphisms, $\Gamma \subseteq SO(3)$, or automorphisms including reflections, $\Gamma \subseteq O(3)$.

(i) Characterize Γ as a representation of a finite group G, for example a group of permuatations. Try to find a suitable group G with the simplest possible definition.

(ii) Is the representation irreducible? Is it absolutly irreducible?

(iii) Determine all isotropy subgroups and their fixed-point spaces.

Extra credit: Consider the octahedron instead of the cube.

Problem 27: Two representations

$$\Gamma_1 : G \to GL(V), \qquad \Gamma_2 : G \to GL(W)$$

of a group G on vector spaces V, W are called equivalent, if there exists an isomorphism $\Phi : V \to W$ such that

$$\forall g \in G \qquad \Phi\Gamma_1(g) = \Gamma_2(g)\Phi.$$

Consider again the irreducible (real and complex) representations of D_n on $\mathbb{C} = \mathbb{R}^2$,

$$\Gamma_k(r)z = \exp(2\pi \mathrm{i} k/n)z, \qquad \Gamma_k(s)z = \overline{z}, \qquad k = 0, \ldots, n-1.$$

Here D_n is generated by the cyclic element s, $s^n = 1$, and the involution r, $r^2 = 1$.

Which of the above representations are complex equivalent? Which of them are real equivalent?

Problem 28: Consider a representation Γ of a group G on a (finite dimensional) vector space V. We can decompose V into subscapes on which G acts irreducibly. Let $\{\Gamma_k; k \in I\}$ be the set of equivalence classes of irreducible representations of G.

The isotypic decomposition of V is given by

$$V = \bigoplus_{k \in I} V_k, \qquad V_k = \bigoplus_{\ell=1}^{n_k} V_{k,l},$$

such that the action of G on $V_{k,l}$ is equivalent to Γ_k. Thus, for each k we find n_k copies of Γ_k.

Now consider an equivariant linaer map $A : V \to V$,

$$\forall g \in G, \quad A\Gamma(g) = \Gamma(g)A.$$

Prove that A is block diagonal with respect to the isotypic decomposition $V = \bigoplus_{k \in I} V_k$.