Homework assignment

Dynamical Systems III

Bernold Fiedler, Stefan Liebscher

http://dynamics.mi.fu-berlin.de/lectures/

due date: Friday, June 10, 2011, 13:00

(As usual, solve (at least) 2 problems, get 1 right.)

Problem 29:

(i) Consider the action of the group $O(n, \mathbb{R}^n)$ of orthogonal matrices on $BC^0(\mathbb{R}^n, \mathbb{R})$,

$$[\gamma u](x) \;:=\; u(\gamma^{-1}x).$$

Prove that the Laplace operator,

$$\Delta \;:\; BC^2(\mathbb{R}^n, \mathbb{R}) \to BC^0(\mathbb{R}^n, \mathbb{R}), \qquad u \mapsto \operatorname{trace} D^2 u,$$

is equivariant, i.e.

$$\forall \gamma \in O(n) \quad \gamma \circ \Delta = \Delta \circ \gamma.$$

(ii) Consider the action of the group $SL(n, \mathbb{R}^n)$ of matrices with derterminant 1 on $BC^0(\mathbb{R}^n, \mathbb{R})$,

$$[\gamma u](x) \;:=\; u(\gamma^{-1}x).$$

Prove that the operator of the Monge-Ampère equation,

$$L \;:\; BC^2(\mathbb{R}^n, \mathbb{R}) \to BC^0(\mathbb{R}^n, \mathbb{R}), \qquad u \mapsto \det D^2 u,$$

is equivariant, i.e.

$$\forall \gamma \in SL(n) \quad \gamma \circ L = L \circ \gamma.$$

Problem 30: Consider a representation of a Lie group Γ on a Hilbert space H. Assume the scalar product on H to be invariant under the group, i.e.

$$\forall\, \gamma \in \Gamma \quad \forall\, u, v \in H \quad \langle \gamma u, \gamma v \rangle_H \;=\; \langle u, v \rangle_H.$$

Let Π be a linear, orthogonal projection $\Pi : H \to V$, i.e.

$$\Pi : H \to H, \qquad \Pi^2 = \Pi, \qquad \forall u \in H \quad \langle \Pi u, (1 - \Pi)u \rangle_H = 0.$$

Assume V to be invariant under the group,

$$\forall\, \gamma \in \Gamma \quad \gamma V \subseteq V.$$

Prove that Π is, both, continuous and equivariant,

$$\forall\, \gamma \in \Gamma \quad \gamma \Pi \;=\; \Pi \gamma.$$

Problem 31: Consider the map

$$f_\lambda(u) = u'' + \lambda \sin u$$

on the space of 2π-periodic functions, $f : \mathbb{R} \times C^2(S^1, \mathbb{R}) \to C^0(S^1, \mathbb{R})$, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$.

(i) Determine the group Γ of linear equivariances of f_λ.

(ii) Characterize the fixed-point spaces of $\sigma \in \Gamma$ resp. $\kappa \in \Gamma$ with

$$[\sigma u](x) = -u(-x), \qquad [\kappa u](x) = u(-x)$$

as subspaces of functions with Dirichlet resp. Neumann boundary conditions.

(iii) Find branches of non-trivial solutions $f_\lambda(u) = 0$ bifurcating from the trivial solution $u \equiv 0$. Which isotropy do the bifurcating solutions possess?

Problem 32: Consider the space

$$V_\ell = \{\ h \in H_\ell(\mathbb{R}^3, \mathbb{R}) \mid \Delta h = 0\ \}$$

of harmonic, homogeneous, scalar polynomials in (x_1, x_2, x_3) of degree ℓ.

(i) Prove that $\Delta : H_\ell \to H_{\ell-2}$ is surjective and thus V_ℓ has dimension $(2\ell + 1)$.

(ii) Prove that

$$[\rho(\gamma)h](x) = h(\gamma^{-1}x)$$

yields a representation of the group $SO(3)$ on V_ℓ.

(iii) Spherical coordinates $(x_1, x_2, x_3) = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ yield

$$\Delta h(x) = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}h\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}h\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}h\right).$$

Spherical harmonics are defined as

$$Y_{\ell,m} = P_{\ell,|m|}(\cos\theta)\exp(im\phi), \qquad -l \le m \le l;$$

$$P_{\ell,m}(z) = (1 - z^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}z^m}P_\ell(z), \qquad P_\ell(z) = \frac{1}{2^\ell \ell!}\frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell}(z^2 - 1)^\ell.$$

Prove that the space V_ℓ is spanned by the real and imaginary parts of the radially scaled spherical harmonics, i.e.

$$V_\ell = \operatorname{span}\left(r^\ell P_{\ell,m}(\cos\theta)\cos(m\phi), r^\ell P_{\ell,m}(\cos\theta)\sin(m\phi), 0 \le m \le l\right).$$