Homework assignment

Dynamical Systems III

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due date: Friday, June 17, 2011, 13:00

(As usual, solve (at least) 2 problems, get 1 right.)

Problem 33: Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function. Consider the Hamiltonian system

$$\dot{x} = f(x) = J\nabla H(x), \qquad J = \begin{pmatrix} 0 & \mathrm{id}_{\mathbb{R}^n} \\ -\mathrm{id}_{\mathbb{R}^n} & 0 \end{pmatrix}.$$

Let the origin, $x = 0$, be an equilibrium, $f(0) = 0$. Let the linearization $Df(0)$ have a pair of algebraically simple eigenvalues $\pm\mathrm{i}$, all other eigenvalues have non-vanishing real parts.

Prove that a neighborhood of $x = 0$ contains a 2-dimensional surface composed of periodic orbits. What do you kow about the periods?

Hint: Consider the system $\dot{x} = g(\lambda, x) := (J + \lambda)\nabla H(x)$ and discuss the resulting Hopf bifurcation.

Problem 34:

(i) Consider a polynomial $p : \mathbb{R}^2 \to \mathbb{R}^2$,

$$p(x,y) \;=\; \sum_{0\le k,\ell;\ k+\ell\le m} p_{k,\ell}\, x^k y^\ell.$$

Prove that p can be written in complex notation, $q : \mathbb{C} \to \mathbb{C}$,

$$p(\mathfrak{Re}\, z, \mathfrak{Im}\, z) \;=\; (\mathfrak{Re}\, q(z), \mathfrak{Im}\, q(z)); \qquad q(z) \;=\; \sum_{0\le k,\ell;\ k+\ell\le m} q_{k,\ell}\, z^k \overline{z}^\ell.$$

(ii) Let Γ_k be linear representations of a group G on Banach spaces X_k, $k = 1, 2$. Consider a smooth equivariant function $\varphi : X_1 \to X_2$,

$$\forall g \in G \quad \forall x_1 \in X_1 \quad \varphi(\Gamma_1(g)x_1) \;=\; \Gamma_2(g)\varphi(x_1).$$

Prove that the Taylor polynomials $T_m\varphi$ to arbitrary order $m \in \mathbb{N}$ are equivariant.

(iii) Let $X_1 = X_2 = \mathbb{R}^2$, and $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be equivariant w.r.t. the the standard action of $SO(2)$. Prove that the Taylor polynomials $T_m\varphi$ to arbitrary order $m \in \mathbb{N}$ have the complex notation

$$T_m\varphi(z) \;=\; z \sum_{0\le k<m/2} \varphi_k |z|^{2k}.$$

Problem 35: [Canards] Consider the van-der-Pol oscillator

$$\begin{aligned} \varepsilon\dot{x} &= y + x - \tfrac{1}{3}x^3, \\ \dot{y} &= \mu - x. \end{aligned}$$

(i) Discuss this singularly perturbed system for fixed μ and $\varepsilon \searrow 0$, as in Dynamics I. Sketch the phase portraits for $\mu > 1$ and $\mu < 1$. Do periodic orbits exist?

(ii) Discuss the bifurcation of the equilibrium $(x, y) = (\mu, \frac{1}{3}\mu^3 - \mu)$ for μ crossing 1 and fixed ε. Which solutions bifurcate?

(iii) How do you reconcile both effect for μ close to 1 and small ε?

Extra credit: Find solutions that link (i) and (ii) in numerical simulations.

Problem 36: The group $SO(3,\mathbb{R})$ of orthogonal (3×3)-matrices with determinant 1 acts on the space $M(3,\mathbb{R})$ of all (3×3)-matrices by conjugation:

$$\rho(\gamma)A := \gamma A \gamma^{-1}, \qquad \gamma \in SO(3,\mathbb{R}).$$

(i) Prove that ρ is indeed a representation.

(ii) Prove that the spaces

$$\begin{aligned} V_0 &= \{\lambda\, \mathrm{id}\ ;\ \lambda \in \mathbb{R}\}, \\ V_1 &= \{A \in M(3,\mathbb{R})\ ;\ A^{\mathrm{T}} = -A\}, \\ V_2 &= \{A \in M(3,\mathbb{R})\ ;\ A^{\mathrm{T}} = A,\ \mathrm{trace}\, A = 0\} \end{aligned}$$

are invarinat, and $M(3,\mathbb{R}) = V_0 \oplus V_1 \oplus V_2$.

(iii) Consider the action of $S^1 = SO(2) \subset SO(3)$ by rotation around the z-axis,

$$\rho(\varphi)A := \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \varphi \in \mathbb{R}/2\pi\mathbb{Z}.$$

Determine the subspaces of V_0, V_1, V_2 on which $SO(2)$ acts irreducibly.

Hint: Consider the Fourier expansion

$$\rho(\varphi)A = \mathfrak{Re} \sum_{k \in \mathbb{Z}} A_k \exp(\mathrm{i}k\varphi)$$

as in problem 25. Which k can yield nonzero coefficients?

(iv) Use (ii) and (iii) to prove that $SO(3)$ act irreducibly on V_0, V_1, V_2.