Homework assignment

Dynamical Systems III

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http://dynamics.mi.fu-berlin.de/lectures/

due date: Friday, June 24, 2011, 13:00

(As usual, solve (at least) 2 problems, get 1 right.)

Problem 37: James Watt's centrifugal governor yields the normalized equation

$$\ddot{\theta} = \Omega \sin 2\theta - \sin\theta - \nu\dot{\theta}.$$

Here, Ω denotes the (normalized) angular velocity, θ denotes the angle of the lever arm, and ν is a friction coefficient.

Which (fixed) values of Ω give rise to a Hopf bifurcation at $(\nu, \theta) = (0, 0)$? Discuss the resulting dynamics and implications for the steam engine.

Problem 38: Consider an interation $x_{n+1} = h(\lambda, x_n)$, $h(\lambda, 0) = 0$, $\lambda \in \mathbb{R}^2$, $x \in \mathbb{R}^N$. Let $\mu(0) = \mathrm{e}^{\pm \mathrm{i}2\pi p/q}$, p, q coprime, be an algebraically simple, non-resonant eigenvalue of the linearization $A(\lambda = 0)$. Let the transversality condition $\det \mu'(0) \neq 0$ hold.

Discuss the bifurcation equation of the subharmonic bifurcation,

$$0 = \mathrm{e}^{-\mathrm{i}\theta}\Phi(\lambda, r\mathrm{e}^{\mathrm{i}\theta}) = \left(\alpha_0(\lambda) + \alpha_1(\lambda)r^2 + \cdots + \alpha_{[q/2]}(\lambda)r^{2[q/2]}\right) r + \beta(\lambda)\mathrm{e}^{-\mathrm{i}q\theta}r^{q-1} + \mathcal{O}(r^{q-1}),$$

for $q \geq 5$, $\beta(0) \neq 0$. Note that $\alpha(\lambda) = \mu(0) - \mu(\lambda) + \mathcal{O}(\lambda)$.

Extra credit: How about $q \leq 4$?

Problem 39: Consider the reversible vector field

$$\dot{x} = f(x) \in \mathbb{R}^N, \qquad f(Rx) = -Rf(x),$$

with linear involution R, $R^2 = \mathrm{id}$.

(i) Prove that every orbit $x(\cdot)$ with $x(0), x(T) \in \mathrm{Fix}(R)$ and $x(0) \neq x(T)$ is periodic. Determine its period. Such orbits are called *reversible periodic orbits.*

(ii) Let $x(\cdot)$ be a reversible periodic orbits. Prove that every Floquet multiplier μ of $x(\cdot)$ is accompanied by Floquet multipliers μ^{-1}, $\overline{\mu}$, and $\overline{\mu}^{-1}$.

Problem 40: Consider the space

$$X = \mathbb{R}^{nq} = \{(x_k)_{k\in\mathbb{Z}}; \forall k\ x_k = x_{k+q} \in \mathbb{R}^n\}$$

and the map

$$f : X \to X, \qquad (f(x))_k = x_{k+1} - h(x_k),$$

for a given map $h : \mathbb{R}^n \to \mathbb{R}^n$. The map

$$S : X \to X, \qquad (Sx)_k = x_{k+1},$$

generates the action of the group $\mathbb{Z}_q$ on X.

(i) Prove that f is equivariant with respect to $\mathbb{Z}_q$.

(ii) Prove that the projection

$$Q : X \to X, \qquad (Qx)_k = \langle e_{(\ell)}, y\rangle e_{(r)} = \left(\frac{1}{q}\sum_{j=0}^{q-1} \overline{\mu}_0^j \overline{y}_{(\ell)}^{\mathrm{T}} x_j\right) \mu_0^k y_{(r)}$$

is equivariant respect to $\mathbb{Z}_q$. Here $\mu_0 = \mathrm{e}^{\pm \mathrm{i}2\pi p/q}$ is the critical eigenvalue of the subharmonic bifurcation as introoduced in class, $y_{(\ell)}, y_{(r)}$ are the associated eigenvectors.