Homework assignment

Dynamical Systems III

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due date: Friday, July 01, 2011, 13:00

(As usual, solve (at least) 2 problems, get 1 right.)

Problem 41: Consider the reversible vector field

$$\dot{x} = f(x) \in \mathbb{R}^N, \qquad f(Rx) = -Rf(x),$$

with linear involution R, $R^2 = \mathrm{id}$.

(i) Prove that every orbit $x(\cdot)$ with $x(0) \in \mathrm{Fix}(R)$, $\alpha(x(\cdot)) = \{x_-\}$ and $x(0) \neq x_- \in \mathrm{Fix}(R)$ is a homoclinic orbit to x_-.

(ii) Let N be an odd number and $\dim \mathrm{Fix}(R) = \frac{1}{2}(N+1)$. Let $x_0 \in \mathrm{Fix}(R)$ be an equilibrium. Prove that under a suitable non-degeneracy condition (which one?) x_0 lies on a curve of equilibria in $\mathrm{Fix}(R)$.

Extra credit: Generalize (ii).

Problem 42: Consider a reversible vector field,

$$\dot{x} = f(x), \qquad Rf(x) = -f(Rx), \qquad x \in \mathbb{R}^N.$$

Let the origin be an equilibrium, $f(0) = 0$. Let the linearization at $x = 0$ be invertible, i.e. all eigenvalues are nonzero.

Prove that then

$$\dim \mathrm{Fix}(R) = N/2.$$

Note: Above assumptions are satisfied for example in the setting of the theorem on reversible Hopf bifurcation. Thus the dimension of the fixed-point space of the reversibility must be half the dimension of the phase space. In particular the dimension of the phase space is even.

Problem 43: Consider the representation of $D_n \times S^1$ on $\mathbb{C} \times \mathbb{C}$ induced by

$$\begin{array}{rcl} \beta(z_1, z_2) & = & (\mathrm{e}^{2\pi \mathrm{i}/n} z_1, \mathrm{e}^{-2\pi \mathrm{i}/n} z_2) \\ \sigma(z_1, z_2) & = & (z_2, z_1) \\ \vartheta(z_1, z_2) & = & (\mathrm{e}^{\mathrm{i}\vartheta} z_1, \mathrm{e}^{\mathrm{i}\vartheta} z_2) \end{array}$$

Determine isotropy subgroups and their fixed-point spaces

Hint: Distinguish cases n mod 4.

Problem 44: [ponies on a merry-go-round] Consider a ring of coupled cells

$$\begin{pmatrix} \dot{x}_k \\ \dot{y}_k \end{pmatrix} = F(x_k, y_k, \lambda) + K(\lambda) \begin{pmatrix} x_{k-1} + x_{k+1} - 2x_k \\ y_{k-1} + y_{k+1} - 2y_k \end{pmatrix} \quad \in \mathbb{R}^2, \qquad k \bmod n,$$

with coupling matrix $K(\lambda)$; or the generalized system

$$\dot{w}_k = g(w_{k-1}, w_k, w_{k+1}, \lambda), \qquad k \bmod n, \qquad w_k \in \mathbb{R}^2, \lambda \in \mathbb{R}$$

with

$$g(u, v, w, \lambda) = g(w, v, u, \lambda).$$

Wich patterns do you expect near a Hopf bifurcation?