# Basic Questions of Dynamical Systems III

1. How is the Lie-derivative of a vector field $f$ with respect to a vector field $g$ defined, $f, g \in \mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N)$? How is the Lie bracket of $f$ and $g$ defined?

2. What is the adjoint representation of a matrix $A$ on the space $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N)$ of vector fields?

3. Formulate a theorem about the normal form of a smooth vector field.

4. What is the normal form of a vector field $f$ at an equilibrium, provided the adjoint representation $\mathrm{ad}_m(A)$ of the linearization has trivial kernel on the space of homogeneous polynomials of degree $m \geq 2$, for every $m$.

5. Formulate the theorem about the normal form of a smooth vector field using the adjoint representation of the transpose of its linearization.

6. Let the linearization $A$ of a vector field $f$ at an equilibrium be normal, $AA^{\mathrm{T}} = A^{\mathrm{T}}A$. Which additional symmetry does the $\mathrm{ad}(A^{\mathrm{T}})$ normal form of $f$ possess, to any finite order?

7. Let the linearization $A$ of a vector field $f$ at an equilibrium be diagonal. What are resonances? How do they relate to the $\mathrm{ad}(A^{\mathrm{T}})$ normal form of $f$?

8. What is a transcritical bifurcation? What is its normal form?

9. What is a Hopf bifurcation? What is its normal form?

10. Formulate a theorem about the normal form of a diffeomorphism.

11. How are the $\mathrm{ad}(A^{\mathrm{T}})$ normal form of a vector field $f = A + \ldots$ and the normal form of the associated time-1-map $\Phi_1$ related?

12. What is a (finite dimensional) Lie group?

13. What is a (finite dimensional) Lie algebra?

14. How is the exponential map of a Lie algebra defined? Is it

    (a) surjective?
    (b) locally surjective?
    (c) injective?
    (d) locally injective?
    (e) a diffeomorphism?
    (f) a local diffeomorphism?

15. When do we call a bounded, linear operator of Banach spaces Fredholm? How is the Fredholm index of such an operator defined?

16. Let $X, Y$ be Banach spaces. Let $L, M : X \to Y$ be bounded linear operators. Which of the following statements are true? How are the Fredholm indices related, in case of true statements?

    (a) If $L, M$ are Fredholm then $L + M$ is Fredholm.
    (b) If $L$ is Fredholm and $M$ is small enough in the operator norm, then $L + M$ is Fredholm.
    (c) If $L$ is small enough in the operator norm then $L$ is Fredholm.
    (d) If $L$ is compact then $L$ is Fredholm.
    (e) If $L$ is Fredholm then $L$ is compact.
    (f) If $L$ is Fredholm and $M$ is compact, then $L + M$ is Fredholm.
    (g) If $X = Y$ and $L$ is Fredholm of index 0 then $L - \mathrm{id}$ is compact.
    (h) If $L$ is compact and Fredholm then $\dim X < \infty$.
    (i) If $L$ is compact and Fredholm then $\dim Y < \infty$.
    (j) If $\dim X < \infty$ and $\dim Y < \infty$ then $L$ is Fredholm.
    (k) If $X = Y$ and $L = \mathrm{id}$ then $L$ is Fredholm.

17. Formulate a theorem on Lyapunov-Schmidt reduction.

18. Formulate the theorem of Crandall and Rabinowitz on local stationary bifurcations.

19. Reformulate the theorem of Crandall and Rabinowitz on local stationary bifurcations in the case of an algebraically simple critical eigenvalue. In particular, state the assumptions as conditions on the parameter dependence of the critical eigenvalue.

20. Draw bifurcation diagrams of possible local stationary bifurcations due to the theorem of Crandall and Rabinowitz.

21. How can we apply the theorem of Crandall and Rabinowitz to the problem $x''(\sigma) + \lambda \sin(x(\sigma)) = 0$, $x'(0) = x'(1) = 0$? (What is a possible functional-analytic setting?)

22. How do we find the bifurcation parameters of the buckled rod $x''(\sigma)+\lambda \sin(x(\sigma)) = 0$, $x'(0) = x'(1) = 0$?

23. What is a local saddle-node bifurcation? Formulate assumptions and sketch a possible phase portrait.

24. What is a (linear) representation of a group $\Gamma$? How are isotropy subgroups of $\Gamma$ defined? How are fixed-point spaces of subgroups $K$ of $\Gamma$ defined? How are invariant subspaces defined?

25. Given a group $\Gamma$ acting linearly on Banach spaces $X, Y$. When do we call a map $F : X \to Y$ $\Gamma$-equivariant?

26. Given a group $\Gamma$ acting linearly on Banach spaces $X, Y$ and an equivariant linear map $L : X \to Y$. Show that Kern $L$ and Range $L$ are $\Gamma$-invariant.

27. Given a group $\Gamma$ acting linearly on $X = \mathbb{R}^N$ and an equivariant linear map $L : X \to X$. Show that the generalized eigenspaces of $L$ are invariant.

28. Formulate a theorem on equivariant Lyapunov-Schmidt reduction.

29. Formulate the theorem of Vanderbauwhede / Cicogna on local equivariant stationary bifurcations.

30. When do we call a linear representation $\rho$ of a group $\Gamma$ on a vectorspace $U$ irreducible? When do we call it absolutely irreducible? Formulate Schur's lemma.

31. What are the irreducible linear representations of $O(2)$, $SO(2)$, $D_n$, $Z_n$, $n \geq 3$?

32. What are the real irreducible linear representations of $O(2)$, $SO(2)$, $D_n$, $Z_n$, $n \geq 3$? Are they absolutely irreducible?

33. What are possible dimensions of irreducible complex representations of Abelian (i.e. commutative) groups?

34. What do we know about the operator $L := \mathrm{id} + \Delta^{-1}$ on $X = L^2(\Omega)$ and therefore about the eigenvalues and eigenspaces of $\Delta, \Delta^{-1}$?

35. Consider the representation of the group $O(n, \mathbb{R}^n)$ of orthogonal matrices on the space $BC^0(\mathbb{R}^n, \mathbb{R})$, given by $[\gamma u](x) := u(\gamma^{-1}x)$. Prove that the Laplace operator $\Delta : BC^0(\mathbb{R}^n, \mathbb{R}) \to BC^2(\mathbb{R}^n, \mathbb{R})$, $u \mapsto \mathrm{trace}\, D^2u$, is equivariant under this representation.

36. Formulate a theorem on local Hopf-bifurcation.

37. How can the Hopf-bifurcation problem be formulated as an equivariant stationary bifurcation problem (to facilitate the application of Lypunov-Schmidt reduction)?

38. Formulate a theorem on local subharmonic bifurcations.

39. When do we call a vector field reversible. What can you say about solutions of reversible systems.

40. Let $\dot{x} = h(x) = Ax$, $A \in M(n, \mathbb{R})$, be a reversible system. How does the reversibility restrict the spectrum of $A$?

41. Formulate a theorem on local Hopf bifurcation in reversible systems.

42. Formulate a theorem on local equivariant Hopf bifurcation.