Homework assignment

Infinite Dimensional Dynamical Systems

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Problem 9: Given two classical solutions $u_1(t,x)$, $u_2(t,x)$ of the same semilinear PDE

$$u_t = u_{xx} + f(x,u,u_x), \qquad 0 < x < L, \qquad \text{+Neumann b.c.},$$

$f \in \mathcal{C}^1$, $t > 0$. Prove or disprove: the zero number $t \mapsto z(u_1(t,\cdot) - u_2(t,\cdot))$ is

(i) upper

(ii) lower

semi-continuous.

Remark: You may assume that the zero number is finite.

Extra credit: Answer the same question for $t \mapsto z(w(t,\cdot)) < \infty$ and arbitrary continuous functions w on $\mathbb{R}^+ \times [0, L]$.

Problem 10: Show the following claim, as Sturm did in 1836! Let

$$\varphi = \sum_{k=m}^{n} a_k \varphi_k \not\equiv 0$$

be a nontrivial linear combination of Sturm-Liouville eigenfunctions,

$$\mu_k \varphi_k = (\varphi_k)_{xx} + b(x)(\varphi_k)_x + c(x)\varphi_k, \qquad \mu_0 > \mu_1 > \cdots$$

Then the zero number is bounded by

$$m \leq z(\varphi) \leq n.$$

Remark: You may use the fact that $z(\varphi_k) = k$.

Problem 11: Let $v \in \mathbb{R}^N$ be a hyperbolic equilibrium of some ODE $\dot{u} = f(u)$ with stable and unstable manifolds W^s, W^u.

(i) Are W^s, W^u transverse at v ?

(ii) Assume $W^s \cap W^u \neq \{v\}$. Show that v possesses a homoclinic orbit $u(t)$, i.e. $u(t) \to v$ for $t \to \pm\infty$. Can W^s and W^u be transverse along $u(t)$?

Problem 12: Consider the Sturm-Liouville eigenvalue problem

$$\mu u_t = u_{xx} + b u_x + c u,$$

with constant real coefficients b, c for 2π-periodic functions, $u(x + 2\pi) = u(x)$.

Obviously, the zero number $z(u(\cdot))$ is even, for $0 \leq x \leq 2\pi$. Why? Determine all eigenvalues μ and eigenfunctions φ as well as (geometric) multiplicities of the eigenvalues. Discuss $z(\varphi)$. Compare this case of periodic b.c. with the case $(\mathcal{N})$ of Neumann b.c.