Homework assignment

Differentialgleichungen I

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http://dynamics.mi.fu-berlin.de/lectures/12SS-Gurevich-Dynamics/

due date: Friday, April 20, 2012, at 10:00am.

Problem 1: Consider the equation $\dot{x} = f(t, x)$. Often we need to schematically draw integral curves.

To construct integral curves it is useful to draw the slopes of those curves on the (t, x) plane. Along the lines $f(t, x) = C$ with constant $C \in \mathbb{R}$ the slope of solutions is the same. Those lines are called **isoclines**. One can sketch the integral curves using the isoclines.

Sketch the integral curves for the following functions

(i) $f(t, x) = t - e^x$,

(ii) $f(t, x) = \frac{x-t}{x^2+1}$.

Problem 2: Write differential equations where the following families of functions are solutions

(i) $x = e^{Ct}$,

(ii) $x = Ct^3$,

(iii) $x = C(t - C)^2$,

(iv) $x = (t - C)^3$,

(v) $x = \sin(t + C)$,

(vi) $Cx = \sin Ct$.

Problem 3: Consider the differential equation

$$-\ddot{x} = \lambda x.$$

(i) Find the first integral for this equation.

(ii) Sketch the phase portrait on the $(x, \dot{x})$ plane.

Compare with pendulum example from class.

Problem 4: Consider the Lotka–Volterra model from class

$$\dot{x} = (\alpha - \beta y)x,$$
$$\dot{y} = (\delta x - \gamma)y$$

with $\alpha, \beta, \gamma, \delta > 0$. Assume that trajectories of solutions don't cross each other in the (x, y) plane.

(i) Show that if $(x(0), y(0))$ is in the first quadrant, then the trajectory $(x(t), y(t))$ stays in the first quadrant for $t \in \mathbb{R}$. This shows that the predator and prey population cannot become negative. Determine what happens if we begin with only predators or only preys.

(ii) Divide the first quadrant of the (x, y) plane into four open regions (see figure below). Show that if $(x(0), y(0)) \in A$, then $(x(t), y(t))$ does the circle $A \to B \to C \to D \to A$.

(iii) Define $H(x, y) = \alpha ln y - \beta y - \delta x - \gamma ln x$. In class we saw that $H(x, y)$ is constant along any trajectory. Use this fact to show that every non-constant solution in the first quadrant is periodic.
(Hint: look at the values of $H(x, y)$ along the line $y = \alpha/\beta$.)