Homework assignment

Differentialgleichungen I

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http://dynamics.mi.fu-berlin.de/lectures/12SS-Gurevich-Dynamics/

due date: Friday, April 27, 2012, at 10:00am.

Problem 5: Consider a differential equation $\dot{x} = f(t,x)$. Assume that the following inequalities hold:

$$f(t,1) < 0, \quad f(t,-1) > 0, \quad t \ge 0. \tag{1}$$

(i) Prove that any solution with initial data $x(0) \in (-1,1)$ satisfies

$$x(t) \in [-1,1], \quad t \ge 0.$$

(ii) Find all solutions of $\dot{x} = \operatorname{sign}(x)|x|^{1/2}$. Note that this equation does not have the property of uniqueness of solution.

(iii) **(Bonus)** Prove or disprove a statement similar to (i), where the equalities in (1) are replaced by

$$f(t,1) \le 0, \quad f(t,-1) \ge 0, \quad t \ge 0.$$

Problem 6: For each of the following equations find:

- The general solution.
- The solution of the Cauchy problem (including the interval of existence of the solution).

(i) $\dot{x} = (1-2t)x^2, \quad x(0) = -\frac{1}{6}$,

(ii) $y' = 2x/(y + x^2 y), \quad y(0) = -2$,

(iii) $\dot{x} = tx^3(1+t^2)^{-1/2}, \quad x(0) = 1$,

(iv) $y' = (3x^2 - e^x)/(2y-5), \quad y(0) = 1$.

Problem 7 (equations reducible to separable equations):
Theoretical background: A differential equation

$$\dot{x} = f(t,x)$$

can often be simplified using a change of variable $u = g(t,x)$. We choose the function g such that x can be expressed as $x = h(u,t)$. Differentiating u gives

$$\dot{u}(t) = g_t(t,x) + g_x(t,x)\dot{x} = g_t(t,h(t,u)) + g_x(t,h(t,u))f(t,h(t,u)).$$

The last term may look more complicated than what we began with, but actually, if g, h are chosen wisely, it gives a significant simplification.

For example, consider an equation of the form

$$\dot{x} = f(at + bx).$$

Set $u = at + bx$, which means that $x = \frac{u-at}{b}$. Plugging it into the formula (try it yourself!) gives

$$\dot{u} = a + b\dot{x} = a + bf(u) := g(u),$$

which is a separable equation.

Problem: Solve the following equations:

(i) Using a change of variables of the form $u = at + bx$ solve the equations

(a) $\dot{x} = -(t+x)^2$,

(b) $\dot{x} = \frac{1}{2t+3x}$,

(c) $\dot{x} = \frac{t+2x}{1+t+2x}$,

(d) $y' = \sqrt{4x + 2y - 1}$.

(ii) Consider a differential equation of the form $\dot{x} = f(t,x)$. If a function f satisfies the equality $f(t,x) = f(\lambda t, \lambda x)$ for any $\lambda \neq 0$, then the differential equation is called *homogeneous*. In that case, the function f can be represented in the form $f(t,x) = F(x/t)$ for $t \neq 0$.

Solve the following homogeneous equations by the change of variables $u = \frac{x}{t}$.

(a) $\dot{x} = \frac{t-x}{t+x}$,

(b) $2t^3\dot{x} = x(2t^2 - x^2)$,

(c) $(t^2 + x^2)\dot{x} = 2tx$.

(iii) Sometimes a differential equation is not homogeneous but can be reduced to a homogeneous one by a change of variables of the form $u = x^\alpha$. Such equations are called *quasihomogeneous*.

Solve the following differential equations.

(a) $2\dot{x} + t = 4\sqrt{x}$,

(b) $2x + (t^2x + 1)t\dot{x} = 0$.

Problem 8: Consider a differential equation of the form

$$\dot{x} = f\left(\frac{at + bx + c}{dt + ex + k}\right),$$

where $a, b, c, d, e, k \in \mathbb{R}$ are some constants. Find changes of variables which reduce this equation to a separable equation in the following cases:

(i) $c = k = 0$.

(ii) $c^2 + k^2 > 0, \quad \det\begin{pmatrix} a & b \\ d & e \end{pmatrix} = 0.$

(iii) $c^2 + k^2 > 0, \quad \det\begin{pmatrix} a & b \\ d & e \end{pmatrix} \neq 0.$ (Hint: use change of variables $u = x - \beta$, $\tau = t - \alpha$.)