Homework assignment

Differentialgleichungen I - Problem Sheet 3

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http://dynamics.mi.fu-berlin.de/lectures/12SS-Gurevich-Dynamics/

due date: Wednesday, May 02, 2012, at 13:00.

General Remark. Often the general solution of a differential equation can be represented in the form

$$F(x,t,C)=0. \tag{1}$$

Formally, according to the lectures, it is not a solution since we do not have a formula for $x(t)$. Moreover (1) doesn't define an implicit solution everywhere, since one can apply the implicit function theorem only when $F_x \neq 0$. At the same time we will consider (1) as a form of writing a solution. In that case we ask you, here and in all future homework, to specify the points where (1) can not be inverted. Note that often in such points the original differential equation have some kind of "degeneracy".

Remember that when we multiply or divide an equation by a function we need to check that it is not zero. Ignoring this effect may cause the appearance of "fake" solutions or disappearance of some solutions.

Problem 9:

(i) In each of the following equations of differentials find the general solution.

(a) $2xy\mathrm{d}x + (x^2+y^2)\mathrm{d}y = 0$,

(b) $e^{-y}\mathrm{d}x - (2y + xe^{-y})\mathrm{d}y = 0$,

(c) $\frac{y}{x}\mathrm{d}x + (y^3 + \ln x)\mathrm{d}y = 0, \quad x>0$,

(d) $3x^2(1+\ln y)\mathrm{d}x = \left(2y - \frac{x^3}{y}\right)\mathrm{d}y$.

(ii) Sometimes the equation

$$A(x,t)\mathrm{d}x + B(x,t)\mathrm{d}t = 0$$

is not an equation of differentials, but there exists a function $M(x,t)$ such that

$$M(x,t)A(x,t)\mathrm{d}x + M(x,t)B(x,t)\mathrm{d}t = 0$$

is an equation of differentials. In that case $M(x,t)$ is called an *integrating factor*.

In each of the following equations find the general solution by integrating factor.

(a) $(3x^2y+2xy+y^3)\mathrm{d}x = -(x^2+y^2)\mathrm{d}y$,

(b) $\dot{x} = e^{2t} + x - 1$,

(c) $\mathrm{d}t + (t/x - \sin x)\mathrm{d}x = 0$,

(d) $\left(3t + \frac{6}{x}\right) + \left(\frac{t^2}{x} + 3\frac{x}{t}\right)\frac{dx}{dt} = 0$.

Problem 10: In each of the following linear equations find the general solution.

(i) $\dot{x} + 3x = t + e^{-2t}$,

(ii) $y' + 2ty = 2te^{-t^2}$,

(iii) $t\dot{x} - x = t^2e^{-t}, \quad t \neq 0$,

(iv) $y' + y = 5\sin 2t$,

(v) $t^3y' = e^{-t} - 4t^2y, \quad t \neq 0$,

(vi) $t\dot{x} + 2x = \sin t, \quad t \neq 0$.

Bonus: Consider equations (iii) and (vi) with $t=0$. Solve for each equation the initial value problem with $x(0)=0$.

Problem 11: (200 points)

(i) The *Bernoulli equation* is a differential equation of the form

$$\dot{x} + p(t)x = q(t)x^n,$$

where $p(t), q(t) : R \to R$ are continuous functions and $n > 1$ is an integer number. It is a nonlinear equation that can be solved analytically. Such equations are famous since there are not too many nonlinear equations that can be solved.

Task: Use a change of variables to reduce the Bernoulli equation to a separable equation or an equation of differentials. (Hint: recall changes of variables which were used in the previous problem sheet.)

(ii) The *Ricatti equation* is a differential equation of the form

$$\dot{x} = p(t)x^2 + q(t)x + r(t),$$

where $p(t), q(t), r(t) : R \to R$ are continuous functions satisfying $p(t), r(t) \neq 0$. In general it is not possible to solve such equations. However, if we assume that one specific solution $x_1(t)$ is known, then the equation can be solved.

Task: Assume that $x_1(t)$ is a given solution to the Ricatti equation. Find a change of variables which reduces the Ricatti equation to the Bernoulli equation.

(iii) In each of the following problems find the general solution.

(a) $y' + 2y = y^2 e^x$,

(b) $y' - 2xy + y^2 = 5 - x^2$ (hint: $y_{1,2}(x) = x \pm 2$ are solutions of this equation),

(c) $3y' + y^2 + \frac{2}{x^2} = 0$.