Homework assignment

Differentialgleichungen I - Problem Sheet 4

Pavel Gurevich, Eyal Ron, Sergey Tikhomirov

http://dynamics.mi.fu-berlin.de/lectures/12SS-Gurevich-Dynamics/

due date: Wednesday, May 09, 2012, at 13:00.

General Remark. In some cases it is convenient to represent a solution of the differential equation

$$F(t, x, \dot{x}) = 0 \tag{1}$$

in a parameterized form, i.e. to introduce a new parameter p and find an expressions for $t = t(p)$, $x = x(p)$. We also consider this form as a "solution" of the equation. In that case $\mathrm{d}x/\mathrm{d}t = \dot{x}(p)/\dot{t}(p)$.

Problem 12: Consider the differential equation (1). Sometimes one cannot derive an explicit formula for $\dot{x}$, but can derive an explicit formula for $x = f(t, \dot{x})$. In that case it is convenient to introduce a new variable $p = \dot{x}$, and take the full differential of the equation $x = f(t, p)$:

$$\mathrm{d}x = f_p(t, p)\mathrm{d}p + f_t(t, p)\mathrm{d}t.$$

Recalling that $\mathrm{d}x = p\mathrm{d}t$, we get a differential equation (written in a form of differentials)

$$p\mathrm{d}t = f_p(t, p)\mathrm{d}p + f_t(t, p)\mathrm{d}t.$$

The latter equation can be handeled by the same methods considered in the previous homework. Note that even if we can't find an explicit solution for p, we can still look for $t(p)$. That way we get a solution parametrized by p. See example in tutorial.

The solution of equation (1) might not be unique (for example $(\dot{x})^2 - x = 0$). A solution $x(t)$ of a differential equation is called **critical** if for each $t \in R$ there exists at least one more solution going through the point $(t, x(t))$ and tangent to the critical one.

It is possible to show that all critical solutions satisfy

$$\frac{\partial F(t, x, \dot{x})}{\partial \dot{x}} = 0.$$

To find a critical solution we need to use this equation and equation (1) in order to eliminate $\dot{x}$, and get a functions for $x(t)$ (probably in a parameterized or implicit form). This function is a "good candidate" for a critical solution. However, we still need to check that it is indeed a solution, and that it is critical. See example in tutorial.

Task: For each of the following equation find a general solution and all of their critical solutions.

(i) $x = \ln(1 + \dot{x}^2)$,

(ii) $x = \dot{x}^2 + 2\dot{x}^3$,

(iii) $\dot{x} = e^{t\dot{x}/x}$.

Problem 13:

(i) A differential equation of the form

$$x = \phi(\dot{x})t + \psi(\dot{x})$$

is called a **Lagrange equation**. A particular case of it is the **Clairaut equation**:

$$x = \dot{x}t + \psi(\dot{x})$$

Task: Using the methods from question 12, find the general solution of the Clairaut equation, note that one of them will have a different form than the others.

(ii) Let $x = w(t, C)$ be a family of solutions for the differential equation

$$F(t, x, \dot{x}) = 0, \tag{2}$$

and let $z(t)$ be another function, which is not a member of $w(t, C)$. If $z(t)$ is tangent at every point $(t, z(t))$ for some member of $w(t, c)$, then it is called **an envelope**.

Task:

(a) Show that if $z(t)$ is an envelope for (2), then it is also a solution of that equation.

(b) Find an envelope for some family of solutions in the Clairaut equation.

Problem 14: If a second order differential equation does not depend on t, i.e,

$$F(x, \dot{x}, \ddot{x}) = 0$$

then one can reduce its order by introducing a new function

$$\dot{x} = p(x). \tag{3}$$

Then

$$\ddot{x} = \frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = \frac{\mathrm{d}p(x)}{\mathrm{d}t} = \frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = p'\dot{x} = p'p$$

and the equation is reduced to a first order equation $F(x, p, p'p) = 0$, where x should be considered now as an independent variable. Remember that after finding $p(x)$ you still need to solve the differential equation (3).

Task: For each of the following equation find the general solution

(i) $\ddot{x} = 2x\dot{x}$,

(ii) $\ddot{x} + \dot{x}^2 = 2e^{-x}$.

Problem 15: Solve either (i) or (ii), but not both (feel free to choose whichever one you want)

(i) For each of the following matrices find the Jordan normal form

(a) $\begin{pmatrix} -10 & 4 \\ -25 & 10 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 0 & 0 \\ 2 & 1 & 3 \\ 5 & 0 & 4 \end{pmatrix}$

For each of the following matrices find the Jordan decomposition, i.e., Jordan normal form and transformation matrix.

(a) $\begin{pmatrix} 5 & -4 \\ 9 & -7 \end{pmatrix}$ (b) $\begin{pmatrix} 9 & 7 & 3 \\ -9 & -7 & -4 \\ 4 & 4 & 4 \end{pmatrix}$

(ii) For the following matrix find the Jordan decomposition, i.e., Jordan normal form and transformation matrix.

$$\begin{pmatrix} 7 & 1 & 2 & 2 \\ 1 & 4 & -1 & -1 \\ -2 & 1 & 5 & -1 \\ 1 & 1 & 2 & 8 \end{pmatrix}$$