Homework assignment

Differentialgleichungen I - Problem Sheet 5

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http://dynamics.mi.fu-berlin.de/lectures/12SS-Gurevich-Dynamics/

due date: Wednesday, May 16, 2012, at 13:00.

Problem 16: For each of the following matrices find the exponent of the matrix.

(i) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Problem 17:

(i) In the following two problems find the solution of the Cauchy problem by two different methods: one is via the general solution (the formula from class with eigenvalues, eigenvectors and associated eigenvectors) and the other is via the exponent of a matrix. Please write the solution in real form without complex valued-functions.

(a) $\begin{matrix} \dot{x} = -2x + y \\ \dot{y} = -5x + 4y \end{matrix}, \quad x(0) = 1, y(0) = 3,$

(b) $\begin{matrix} \dot{x} = -3x + 2y \\ \dot{y} = -x - y \end{matrix}, \quad x(0) = 1, y(0) = 2.$

(ii) In the following two problems find the general solution. In problem (a) the general solution should be real-valued.

(a) $\begin{matrix} \dot{x} = 5x - 3y - 2z \\ \dot{y} = 8x - 5y - 4z \\ \dot{z} = -4x + 3y + 3z \end{matrix},$

(b) $\dot{z} = (3 + 2i)z.$

Problem 18:

(i) Construct matrices A, B such that $e^{A+B} \neq e^A e^B$ and $e^A e^B \neq e^B e^A$.

(ii) Consider $(N \times N)$-matrices A, B and the differential equations

$$\dot{x} = Ax \tag{1}$$

$$\dot{y} = By. \tag{2}$$

Let $x(t) = \phi(t, x_0)$ be the solution of the Cauchy problem $x(0) = x_0$ of (1). Let $y(t) = \psi(t, y_0)$ be the solution of the Cauchy problem $y(0) = y_0$ of (2). Note that

$$\phi(t, x_0) = e^{At} x_0, \quad \psi(t, y_0) = e^{Bt} y_0.$$

We say that equations (1) and (2) *commute* if

$$\phi(t_1, \psi(t_2, x)) = \psi(t_2, \phi(t_1, x)), \quad \text{for any } t_1, t_2 \in R,\ x \in R^N.$$

Prove that equations (1) and (2) commute if and only if $AB = BA$.

(Hint: consider $t_1 = t_2 = t$ and let $t \to 0$).

Problem 19: The following problems are optional. It is an opportunity to improve the grades of problems 7, 9 and 10. The grades of the problems below will replace the original grades of the corresponding problems for those who will choose to do it. It is highly recommended for students who got less that 60 points in one of the aforementioned problems.

- **Problem 7 (100 points).** In each of the following differential equations find the general solution.

  (i) $y' = 10^{x+y}$,

  (ii) $y' = \cos(y - x)$,

  (iii) $y^2 + x^2 y' = xyy'$,

  (iv) $xy' = y \cos \ln \frac{y}{x}$,

  (v) $(y' + 1) \ln \frac{y+x}{x+3} = \frac{y+x}{x+3}$. (Hint: remember problem 8),

  (vi) $y' = y^2 - 2/x^2$.

- **Problem 9 (100 points).** In each of the following differential equations find the general solution.

  (i) $(2 - 9xy^2)x\mathrm{d}x + (4y^2 - 6x^3)y\mathrm{d}y = 0$,

  (ii) $(x^2 + y^2 + x)\mathrm{d}x + y\mathrm{d}y = 0$,

  (iii) $xy^2(xy' + y) = 1$,

  (iv) $-y^2\mathrm{d}x + (e^x + 2y)\mathrm{d}y = 0$.

- **Problem 10 (100 points).** In each of the following differential equations find the general solution.

  (i) $xy' - 2y = 2x^4$,

  (ii) $x^2 y' + xy + 1 = 0$,

  (iii) $(xy' - 1) \ln x = 2y$,

  (iv) $y' + 2y = y^2 e^x$.