Homework assignment

Differentialgleichungen I - Problem Sheet 6

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http://dynamics.mi.fu-berlin.de/lectures/12SS-Gurevich-Dynamics/

due date: Wednesday, May 23, 2012, at 13:00.

Problem 20: Solve the following differential equations using the characteristic polynomial. In some of the problems you will need to make a change of variables first.

(i) $y^{(4)} + 2y'' + y = 0$,

(ii) $y^{(4)} - y = 0$,

(iii) $x^2y'' - 4xy' + 6y = 0$,

(iv) $x^2y''' = 2y'$.

Problem 21:

(i) Consider the linear system of differential equations

$$\dot{x} = A(t)x, \tag{1}$$

where $A(t) \in \mathbb{R}^{n\times n}$. Let $\phi(t) = (\phi_1(t), ..., \phi_n(t))^T$, $t \in [t_1, t_2]$, be a known solution of (1). Assume that $\phi_1(t) \neq 0$ for $t \in [t_1, t_2]$.
Show that there exists a unique $u(t) \in \mathbb{R}$ such that the following change of variables

$$x(t) = u(t)\phi(t) + y(t),$$

where $y(t) = (0, y_2(t), .., y_n(t))^T$, changes (1) to a $(n-1)$-dimensional system in the $(y_2(t), ..., y_n(t))$ variables. This change of variables is called the *reduction of order* method.

(ii) In each of the following problems use the reduction of order method to find the general solution.

(a)

$$\begin{matrix} x' = 3y \\ y' = \frac{2y}{t} - \frac{2x}{3t^2} \end{matrix}, \quad t > 0, \quad \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} 3t \\ 1 \end{pmatrix},$$

(b)

$$\begin{matrix} e^t x' + e^t x - y = 0 \\ (e^t+1)y' - 2y - e^{2t}x = 0 \end{matrix}, \quad t > 0, \quad \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} 1 - e^{-t} \\ e^t \end{pmatrix}.$$

Problem 22: Consider the differential equation

$$\ddot{x} + x = f(t)$$

(i) Find the general solution of the equation above.

(ii) Find sufficient estimates on $|f(t)|$ that guarantee that all the solutions of this equation are bounded as $t \to +\infty$.

(Hint: reduce the equation to a system.)

Problem 23: Consider the linear system of differential equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ x(0) &= x_0, \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ and A is an $n \times n$ constant matrix. Let $\phi(t, x_0)$ be the solution of (2). Take K_0 to be a compact subset of $\mathbb{R}^n$. Denote

$$K_t = \{\phi(t, x_0) \in \mathbb{R}^n, \text{ such that } x_0 \in K_0\}.$$

and

$$V(t) = Volume\ K_t = \int_{K_t} dx$$

Prove:

(i) $\dot{V}(t) = trA \cdot V(t)$. (Hint: find $\dot{V}(t)$ by looking at $\lim_{s \to 0} \frac{V(t+s) - V(t)}{s}$ and take Liouville's formula into account.)

(ii) For which conditions on A there is *volume preservation*, i.e, $V(t) \equiv V(0)$.