Homework assignment

Differentialgleichungen I - Problem Sheet 7

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http://dynamics.mi.fu-berlin.de/lectures/12SS-Gurevich-Dynamics/

due date: Wednesday, May 30, 2012, at 13:00.

Problem 24:

(i) **(10 points)** Prove that every n-order linear differential equation of the form

$$x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \dots + a_n(t)x(t) = 0 \tag{1}$$

$$x^{(n-1)}(t_0) = x_0^{n-1}, \dots, x(t_0) = x_0^0. \tag{2}$$

can be reduced to an n-order linear system of the form

$$y'(t) = A(t)y(t), \tag{3}$$

$$y(0) = y_0. \tag{4}$$

(ii) **(45 points)** Show that equation (1) has exactly n linearly independent solutions.

(iii) **(45 points)** Let $\{x_i\}_{i=1}^n$ be a sequence of n linearliy independent solutions of (1). Define the Wronskian $W(t)$ of such a sequence to be

$$W(t) = det \begin{pmatrix} x_1 & \dots & x_n \\ \dot{x}_1 & \dots & \dot{x}_n \\ \dots & \dots & \dots \\ x_1^{(n-1)} & \dots & x_n^{(n-1)} \end{pmatrix}.$$

Prove Liouville's formula for higher order equations, i.e.

$$W(t) = W(t_0)e^{-\int_{t_0}^t a_1(s)ds}.$$

Problem 25 (method of undetermined coefficients): In order to solve nonhomogeneous linear equations, we considered in class the general method of variation of constants. For certain right-hand side a simpler method, *the method of undetermined coefficients*, can be used.

(i) **(10 points)** Consider the equation

$$L(D)z = F(t), \tag{5}$$

where L is an n-order polynomial, and $F(t)$ is a continuous function for all $t \in \mathbb{R}$. Let z_p be a paticular solution of (5), and $u(t)$ be a general solution of the homogenous equation $L(D)u = 0$. Show that $z = z_p + u$ is a general solution of (5).

(ii) **(90 points)** If the right-hand side of equation (5) has the form

$$P_m(t)e^{\gamma t},$$

where P_m is a polynomial of degree m, then a particular solution can be found in the form

$$x(t) = t^s Q_m(t) e^{\gamma t},$$

where Q_m is a polynomial of degree m, and s is equal to the multiplicity of the root γ of the characteristic polynomial of the homogenious equation (if γ is not a root then $s = 0$). If the right-hand side of the form

$$e^{\gamma t}(P(t)\cos(\beta t) + Q(t)\sin(\beta t)),$$

then a particular solution can be found in the form

$$x(t) = t^s e^{\gamma t}(R_m(t)\cos(\beta t) + T_m(t)\sin(\beta t)),$$

where R_m, T_m are polynomials of degree m, and s is determined as above.

Task: In each of the following problems find the general solution.

(a) $y'' - 9y = e^{3x}\cos x$,

(b) $y'' - 5y' = 3x^2 + \sin 5x$,

(c) $x^2 y'' - 2y = \sin \ln x$.

Problem 26: (150 points)

(i) Consider the differential equation

$$\dot{x} = ax + f(t),$$

where $a > 0$, and $-1 \leq f(t) \leq 1$.

Prove that there exists exactly one solution bounded on the whole real line.

(ii) Consider the differential equation

$$\dot{x} = Ax + f(t),$$

where A is an $n \times n$ matrix, x is an n-dimensional vector, $|f(t)| \leq 1$, and

(a) All eigenvalues of A have positive real part.

(b) **(bonus)** All eigenvalues of A have non zero real part.

Prove that there exists exactly one solution bounded on the whole real line.

(iii) Consider the differential equation

$$\dot{x} = A(t)x + f(t), \tag{6}$$

where $A(t)$ is an $n \times n$ matrix, $A(t + \tau) = A(t)$, and $|f(t)| \leq 1$. Let C be the monodromy matrix of some fundamental solution of the homogeneous equation $\dot{x} = A(t)x$. Prove that if all eigenvalues of C are bigger than 1, then (6) has exactly one solution bounded on the whole real line.

Problem 27: **(50 points)** Find the monodromy matrix for the following system of equations.

$$x' = \sin(t)x - y$$
$$y' = x + \sin(t)y.$$