Homework assignment

Differentialgleichungen I - Problem Sheet 9

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http://dynamics.mi.fu-berlin.de/lectures/12SS-Gurevich-Dynamics/

due date: Wednesday, June 13, 2012, at 13:00.

Problem 30: Consider a differential equation of the form

$$\dot{x} = x^n + p_1(t)x^{n-1} + \cdots + p_n(t), \tag{1}$$

where $x \in R$, $n \in \mathbb{N}$ and $p_k \in C$ are ω-periodic functions for some $\omega > 0$.

(i) Prove that if $n = 1$ then (1) has exactly one ω-periodic solution.

(ii) Prove that if $n = 2$ then (1) has at most two ω-periodic solutions.

Problem 31:

(i) Prove that if $f(t, x) : \mathbb{R} \times \mathbb{R}^n$ is C^1 then f is Lipschitz continuous with respect to x on any bounded set.

(ii) Construct a function, $f(t, x)$, that is continuous on $[-1, 1] \times [-1, 1]$ and is C^1 with respect to $x \in [-1, 1]$ for every $t \in [-1, 1]$, but is not Lipschitz with respect to x on $[-1, 1] \times [-1, 1]$.

(iii) Consider the differential equation

$$x = t\dot{x} + (\dot{x})^2. \tag{2}$$

Note that $x_1 = -2t + 4$ and $x_2 = -\frac{t^2}{4}$ are both solutions of (2). Note as well that all the expressions appearing in (2) are smooth functions of the dependent and independent variables. Explain why is this not a contradiction to theorem of uniqueness of the solution of a differential equation.

Problem 32: Consider the differential equation

$$x' = f(t, x),$$

where $f \in C^0$.

(i) Consider $x \in R$ and assume that

$$(x - y)(f(x, t) - f(y, t)) \leq 0$$

for any $x, y \in R, \ t \geq 0$.

(ii) Consider $x \in R^n$ and assume that

$$(x - y, f(x, t) - f(y, t)) \leq 0,$$

for $x, y \in R^n$ and $t \geq 0$. (Here $(\cdot, \cdot)$ means the scalar product).

Prove that there exists at most one solution of the initial value problem $x(0) = x_0$ for $t \geq t_0$.