Homework assignment

Differentialgleichungen I - Problem Sheet 10

Pavel Gurevich, Eyal Ron, Sergey Tikhomirov

http://dynamics.mi.fu-berlin.de/lectures/12SS-Gurevich-Dynamics/

due date: Wednesday, June 20, 2012, at 13:00.

Problem 33: Let $K(t,s) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f(t) : \mathbb{R} \to \mathbb{R}$ be two continuous bounded functions. Prove that there exists $T > 0$ such that for any $x_0 \in R$ there exists a unique solution to the following equation

$$x(t) = f(t) + \int_0^t K(t,s)x(s)ds$$

for $|t| \leq T$, satisfying $x(0) = x_0$.

Hint: Consider using method of iterations.
Remark: Such equations are called linear Volterra integral equation of the second kind.

Problem 34: Consider the function $f : R^2 \to R$ defined as follows

$$f(x,y) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ y \geq 2x, \\ \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ y < 2x. \end{cases}$$

Note that f is discontinuous. Prove that the differential equation

$$\frac{\partial}{\partial t} \begin{pmatrix} x \\ y \end{pmatrix} = f(x,y)$$

has no solution for $t > 0$ of the initial value problem $x(0) = y(0) = 0$.

Problem 35: Consider the differential equation

$$\dot{x} = f(t, x), \tag{1}$$

where $x \in R$, $f \in C^1$, $f(t+1, x) = f(t, x)$

(i) Prove that for any solution $x(t)$ of (1) there exists $C > 0$ such that either

$$x(t) < C \quad \text{for all} \quad t > 0$$

or

$$x(t) > -C \quad \text{for all} \quad t > 0.$$

(ii) Assume that for some solution $x(t)$ of (1) there exists $C > 0$ such that $|x(t)| < C$ for all $t > 0$. Prove that

$$\lim_{t \to +\infty} x(t+1) - x(t) = 0.$$

Hint: Use that any invertible continuous function of the real line into itself is monotone.

Problem 36: **(bonus)** Consider a differential equation of the form

$$\dot{x} = x^n + p_1(t)x^{n-1} + \cdots + p_n(t), \tag{2}$$

where $x \in R$, $n \in \mathbb{N}$ and $p_k \in C$ are ω-periodic functions for some $\omega > 0$. Prove or disprove that for any $n > 2$, equation (2) has a finite number of ω-periodic solutions.