Homework assignment

Differentialgleichungen II

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http://dynamics.mi.fu-berlin.de/lectures/12WS-Gurevich-PDE/

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For a function $u : R^n \to R$, $u \in C^2$ we denote $\Delta u(x) = \sum_{i=1}^n u_{x_i x_i}(x)$.

Problem 1: (50 points) Determine the type of the following equations (linear, quasi-linear, fully nonlinear)

(i) $\Delta u = 0$.

(ii) $u_{x_1} + u_{x_2x_2x_2x_2} = 0$.

(iii) $u_t + uu_x = f(u)$.

(iv) $u_t - \Delta(u^2) = 0$.

(v) $\text{div}((|\nabla u|)^{p-2}\nabla u), \quad p \geq 2$.

Problem 2: (100 points) Recall the integration by parts formula. Let $u, v : R^n \to R$, $u, v \in C^1(\bar{Q})$, $Q \subset R^n$ an open set with C^1 boundary ∂Q. Then

$$\int_Q u_{x_i} v dx = -\int_Q u v_{x_i} dx + \int_{\partial Q} uv\nu_i dS,$$

where ν is the outer unit normal on the boundary.

Prove the following formulas for $u, v \in C^2(\bar{Q})$:

(i)

$$\int_Q \Delta u dx = \int_{\partial Q} \frac{\partial u}{\partial \nu} dS.$$

(ii)

$$\int_Q v\Delta u dx = \int_{\partial Q} \frac{\partial u}{\partial \nu} v dS - \int_Q \nabla v \nabla u dx.$$

(iii)

$$\int_U u\Delta v - v\Delta u dx = \int_{\delta U} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}.$$

Problem 3: (150 points)

(i) Solve the following equation using characteristics

$$x_1 u_{x_1} + x_2 u_{x_2} = 2u, \quad u(x_1, 1) = g(x_1),$$

for $x \in Q$, where $Q = \{x_2 \geq 1\}$

(ii) Write the characteristics ODE for the following PDE

(a)

$$2u_t = u_x + xu$$

(b)

$$u_t + (1 + x^2)u_x - u = 0.$$