Homework assignment

Differentialgleichungen II

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http://dynamics.mi.fu-berlin.de/lectures/12WS-Gurevich-PDE/

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Problem 7: Consider the following problem

$$F(x, u, \nabla u) = 0, \quad x \in Q,$$

$$u(x) = g(x), \quad x \in \Gamma,$$

where $Q \subset \mathbb{R}^n$ is a domain with boundary Γ. Assume that Γ is a hypersurface defined in a neighborhood of a point $x^* \in \Gamma$ by

$$x_n = \varphi(x_1, \dots, x_{n-1}),$$

where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, $\varphi \in C^\infty$.

Recall that, in the lectures, the compatability and noncharacteristic conditions were introduced for a flat boundary. Obtain such compatability and noncharacteristic conditions in terms of F, g and φ.

Problem 8: (Transformation of quasilinear equations) Consider the quasilinear problem

$$a(x, u) \cdot \nabla u = g(x, u) \text{ for } x \in U \subset \mathbb{R}^n. \tag{1}$$

(i) Consider a change of coordinates $y = \Phi(x), x = \Psi(y)$ with a bijective mapping $\Psi : V \to U$, where $\Phi, \Psi \in C^1$. Let $v(y) = u(\Psi(y)) = (u \circ \Psi)(y)$. Which equation holds for v, if u is a solution of (1)?

(ii) Consider a bijection $\alpha : \mathbb{R} \to \mathbb{R}$, such that $\alpha, \alpha^{-1} \in C^1$. Which equation holds for $w(x) := \alpha(u(x))$, if u is a solution of (1)?

(iii) Consider the case

$$\partial_t u + \tilde{a}(t, x, u) \cdot \nabla_x u = b(t, x, u).$$

Give the equations for v and w from (i) and (ii) in the form $\partial_t v + ...$ and $\partial_t w + ...$ respectively. Here the coordinate change Ψ in (i) should not depend on t.

(iv) Give bijections Ψ and α that transform the equation

$$\partial_t u + \partial_{x_1} u + x_1 \partial_{x_2} u = u,$$

into the equation

$$\partial_t w + \partial_{x_1} w = 1.$$

Construct the general solution w and provide a formula for the general solution u.

Problem 9: Let $a, b \in \mathbb{R}$ and consider the equation

$$u_t + au_x = bu \text{ for } t > 0 \text{ and } x \in (0, \infty) \tag{2}$$

together with the Cauchy initial condition $u(0, x) = g(x)$.

(i) Show that for $a \leq 0$ there is a unique solution for (2) (recall that $t > 0$).

(ii) Provide all the solutions of (2) for the case $a > 0$. Confirm local uniqueness near $(t, x) = (0, x^*)$ and show global uniqueness.

Problem 10: Create a PDE in the unit disk with boundary condition on the unit circle such that a C^2 solution exists everywhere in the disc except for the origin. Motivate your choice.