Homework assignment

Differentialgleichungen II

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http://dynamics.mi.fu-berlin.de/lectures/12WS-Gurevich-PDE/

due date: 14:00, Wednesday, November 14, 2012

Problem 11:

(i) Show that $\Gamma = \{x \in \mathbb{R}^2 : x_2 = 0\}$ is a characteristic line for the equation

$$u_{x_1x_2} = f(x).$$

(ii) Consider the following problem:

$$\begin{cases} u_{x_1x_2} = f(x), \qquad x \in B, \\ u|_{B\cap\{x_2=0\}} = u_0(x_1), \\ u_{x_2}|_{B\cap\{x_2=0\}} = u_1(x_1), \end{cases} \tag{1}$$

where B is the unit ball centered at the origin, $f \in C(\mathbb{R}^2)$, $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$. Prove that (1) has a solution $u \in C^2(B)$ if and only if

$$\frac{du_1(x_1)}{dx_1} \equiv f(x_1, 0). \tag{2}$$

In addition, prove that if (2) holds, then all solutions are given by

$$u(x_1, x_2) = \int_0^{x_1} d\xi_1 \int_0^{x_2} f(\xi_1, \xi_2)d\xi_2 + u_0(x_1) + g(x_2),$$

where g is an arbitrary C^2 function satisfying

$$g(0) = 0, \quad \frac{dg(0)}{dx_2} = u_1(0).$$

Problem 12: **[200 points]** Consider the equation

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + f(x,y,u,u_x,u_y) = 0, \qquad (x,y) \in Q \subset \mathbb{R}^2, \quad (3)$$

where f represents lower-order terms, $a(x,y) \neq 0$ in Q.

(i) Denote $\Delta(x,y) = \Delta = b^2 - ac$. Show that (3) (at a point $(x,y) \in Q$) is

(a) hyperbolic if $\Delta(x,y) > 0$,

(b) parabolic if $\Delta(x,y) = 0$,

(c) elliptic if $\Delta(x,y) < 0$.

(ii) Introduce new variables

$$\begin{cases} \xi = \xi(x,y), \\ \eta = \eta(x,y), \end{cases} \qquad (4)$$

where $\xi, \eta \in C^2(Q)$, $D = det \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix} \neq 0$ in Q. Consider a function $v(\xi,\eta)$ such that $u(x,y) = v(\xi(x,y), \eta(x,y))$. Then $v(\xi,\eta)$ satisfies

$$Av_{\xi\xi} + 2Bv_{\xi\eta} + Cv_{\eta\eta} + F(\xi,\eta,v,v_\xi,v_\eta).$$

Prove that

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2, \qquad C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2,$$

and find an expression for B. Prove that $B^2 - AC = (b^2 - ac)D^2$.

(iii) Let $\Delta(x,y) > 0$ in Q, and suppose that the 1st-order PDEs

$$\begin{aligned} a\xi_x + (b + \sqrt{b^2 - ac})\xi_y &= 0, \\ a\eta_x + (b + \sqrt{b^2 - ac})\eta_y &= 0 \end{aligned} \qquad (5)$$

have solutions $\xi, \eta \in C^1(Q)$ such that $\xi_y \neq 0, \eta_y \neq 0$ in Q. Show that the lines $\xi(x,y) = c_1, \eta(x,y) = c_2$ are characteristic lines for equation (3).

(iv) Find a nondegenerate change of varialbes which reduces (3) to the canonical form in the whole domain Q. (Hint: first make the change of variables (4), where $\xi(x,y), \eta(x,y)$ are solutions of (5). Make sure that it is nondegenerate.)

(v) Let $\Delta(x,y) = 0$ in Q. Assume the equation $a\xi_x + b\xi_y = 0$ has a solution $\xi \in C^1(Q)$. Take an arbitrary $\eta(x,y)$ such that $det \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix} \neq 0$ in Q. Show that the change of variables $\xi = \xi(x,y)$, $\eta = \eta(x,y)$ reduces (3) to the canonical form.

(vi) Let $\Delta(x,y) < 0$. Let

$$\begin{aligned} \xi(x,y) &= \phi(x,y) + i\psi(x,y), \\ \eta(x,y) &= \phi(x,y) - i\psi(x,y) \end{aligned}$$

be solutions of (5), where $\phi, \psi \in C^1(Q)$ are real-valued. Show that the change of variables $\xi = \phi(x,y)$, $\eta = \psi(x,y)$ reduces (3) to the canonical form.

Problem 13:

(i) Reduce the following equation to the canonical form, using the method from class (by a linear change of variables):

$$u_{xx} - 2u_{xy} - 3u_{yy} + u_y = 0.$$

(ii) In each of the following equations

- Indicate the domain in which the equation is hyperbolic/parabolic/elliptic.
- Reduce to the canonical form in each of the above domains by the method from Problem 12

(a) $u_{xx} - xu_{yy} = 0.$

(b) $xu_{xx} - yu_{yy} = 0.$

Problem 14:

(i) Prove that the second order PDE

$$\sum a_{ij}(x)u_{x_i x_j} + \text{lower-order terms} = f(x), \quad x \in Q \subset \mathbb{R}^n, \tag{6}$$

is elliptic at a point $x \in Q$ if and only if there are two constants $c_1, c_2 > 0$ such that

$$c_1|x_i|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq c_2|\xi|^2$$

for all vectors $\xi \in \mathbb{R}^n$.

(ii) Prove that if the PDE is elliptic in the closure $\overline{Q}$, then c_1 and c_2 can be chosen not depending on $x \in \overline{Q}$.