Homework assignment

Differentialgleichungen II

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http://dynamics.mi.fu-berlin.de/lectures/12WS-Gurevich-PDE/

due date: 14:00, Tuesday-Wednesday, November 27-28, 2012

Problem 19: Find the solutions of the following initial boundary-value problems

(i) $u_{xx} = u_{tt}$, $0 < x < 1$, $u|_{x=0} = t + 1$, $u|_{x=1} = t^3 + 2$, $u|_{t=0} = x + 1$, $u_t|_{t=0} = 0$.

(ii) $u_t = u_{xx} + u - x + 2 \sin 2x \cos x$, $0 < x < \pi/2$, $u|_{x=0} = 0$, $u_x|_{x=\pi/2} = 1$, $u|_{t=0} = x$.

Problem 20:

(i) Consider the following spectral problem

$$\Delta u(x, y) = -\lambda u(x, y), \quad x \in (0, K), y \in (0, L), \tag{1}$$

where $K, L > 0$, $\lambda \in R$, with the following boundary conditions

(a) $u|_{x=0} = u|_{x=K} = 0$, $u|_{y=0} = u|_{y=L} = 0$,

(b) $u_x|_{x=0} = u_x|_{x=K} = 0$, $u_y|_{y=0} = u_y|_{y=L} = 0$.

Find all eigenfunctions (and corresponding eigenvalues) of the form

$$u(x, y) = f(x)g(y). \tag{2}$$

Bonus: Prove that all eigenfunctions of the above spectral problem have the form (2).

(ii) Find the solutions of the following initial boundary-value problems

(a)

$$u_{tt} = \Delta u \quad (0 < x < \pi,\ 0 < y < \pi),$$
$$u|_{x=0} = u|_{x=\pi} = 0,\ u|_{y=0} = u|_{y=\pi} = 0,$$
$$u|_{t=0} = 3 \sin x \sin 2y, \quad u_t|_{t=0} = 5 \sin 3x \sin 4y.$$

(b)

$$u_t = \Delta u \quad (0 < x < \pi,\ 0 < y < 2\pi),$$
$$u_x|_{x=0} = u_x|_{x=\pi} = 0\ u_y|_{y=0} = u_y|_{y=2\pi} = 0,$$
$$u|_{x=0} = 2 \cos x(\cos y/2 + 1)^2.$$

Problem 21:

(i) Prove that the Schwartz space is dense in

(a) $L_1(R^n)$ (in the topology of $L_1(R^n)$),

(b) $L_2(R^n)$ (in the topology of $L_2(R^n)$).

(ii) Let $f \in L_2(R^n)$. Denote $B_N = \{x \in R^n : |x| < N\}$ and

$$F_N(\xi) = (2\pi)^{-n/2} \int_{B_N} e^{-ix\xi} f(x)dx.$$

Prove that the integral in the righthand side converges absolutely for any $\xi \in R^n$, defines a function $F_N(\xi)$ from $L_2(R^n)$ and

$$\|\hat{f} - F_N\|_{L_2(R^n)} \to_{N\to\infty} 0.$$

Problem 22:

(i) Prove or disprove that if $f \in L_1(R^n)$ then $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

(ii) Let $\phi(x) = e^{-|x|^2/2}$, $x \in R^n$. Prove that $\hat{\phi}(\xi) = \phi(\xi)$.

Hint: Use that the integral of a complex analytic function along a closed contour equals 0.