Homework assignment

Differentialgleichungen II

Pavel Gurevich, Sergey Tikhomirov

http://dynamics.mi.fu-berlin.de/lectures/12WS-Gurevich-PDE/

due date: 14:00, Tuesday, December 4, 2012

Problem 23:

(i) Consider the equation

$$\Delta u(x) = f(x), \quad x \in Q,$$

where $Q = \{x \in \mathbb{R}^2 : |x| < 1\}$. Write a partial differential equation for the function $v(r, \varphi) = u(r\cos\varphi, r\sin\varphi)$.

(ii) Consider the equation

$$\Delta u(x) = f(|x|), \quad x \in Q,$$

where $Q = \{x \in \mathbb{R}^n : |x| < 1\}$. Assume that $u(x) = v(|x|)$, where v is a real-valued function. Write an ordinary differential equation for the function v.

Problem 24:

(i) Find all the eigenvalues and all the corresponding eigenfunctions of the following spectral problem

$$-e''(x) = \lambda e(x), \quad x \in (0, l), \tag{1}$$
$$e(0) = e(l), \quad e'(0) = e'(l),$$

where $l = 2\pi$.

(ii) Find a solution of

$$\Delta u(x) = 0, \quad x \in Q,$$

and

(a) $Q = \{x \in \mathbb{R}^2 : |x| < 1\}$

$$u|_{\partial Q} = g(\varphi),$$

(b) $Q = \{x \in \mathbb{R}^2 : 1/2 < |x| < 2\}$

$$u|_{|x|=1/2} = g_1(\varphi),$$

$$u|_{|x|=2} = g_2(\varphi),$$

as a formal series

$$u = \sum_{j=1}^{\infty} u_j(r) e_j(\varphi),$$

where φ is a polar angle and g, g_1, g_2 are real-valued functions
Hint: Use (i). Note that in (a) the function should be bounded at 0.

Problem 25:

(i) Find a function $f \in L_2(Q)$, $Q = \{x \in \mathbb{R}^2 : |x| < 1\}$ such that the generalized derivatives (from the space $L_{2,loc}(Q)$) f_{x_1} and f_{x_2} do not exist, but the generalized derivative $f_{x_1 x_2}$ exists.

(ii) In each of the following problems determine if the following generalized derivative (from the space $L_{2,loc}(Q)$) exists, and if yes find it.

(a) $\frac{\partial^2}{\partial x_1 \partial x_2} f(x)$, where $f(x) = |x_1 x_2|$ in $Q = \{x \in \mathbb{R}^2 : \ |x| < 1\}$.

(b) $\frac{\partial}{\partial x} f(x)$, where $f(x) = \sqrt{|x|}$ in $Q = \{x \in \mathbb{R}, |x| < 1\}$.

(c) $\frac{\partial}{\partial x} f(x)$ and $\frac{\partial^2}{\partial x^2} f(x)$, where

$$f(x) = \begin{cases} 0, & x < 0 \\ \sin x & x \geq 0, \end{cases}$$

in $Q = \{x \in \mathbb{R}, |x| < 1\}$.

Problem 26: Let Q be a domain in $\mathbb{R}^n$.

(i) Let $a \in C^k(\bar{Q})$ and let $|D^\alpha a(x)|$ be bounded for all $|\alpha| \leq k$. Prove that for any $f \in H^k(Q)$

(a) $(af)_{x_i} = a_{x_i} f + a f_{x_i}$, where a_{x_i} is a classical derivative, $(af)_{x_i}$ and f_{x_i} are generalized derivatives.

(b) $af \in H^k(Q)$ and $\|af\|_{H^k(Q)} \leq C\|f\|_{H^k(Q)}$, where $C > 0$ does not depend on f.

(ii) Let $|\beta| \leq k$. Prove that

$$D^\beta : H^k(Q) \to H^{k-|\beta|}(Q)$$

is a bounded linear operator.

(iii) Let $f \in H^k(Q)$ and let f_h be its mollifier ($h > 0$). Prove that

$$\|f_h - f\|_{H^k(Q')} \to 0, \quad \text{as } h \to 0,$$

for any $Q' \Subset Q$.