Homework assignment

Differentialgleichungen II

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http://dynamics.mi.fu-berlin.de/lectures/12WS-Gurevich-PDE/

due date: 14:00, Tuesday, December 4, 2012

Problem 23: Prove or disprove that the embedding $H^1(R) \subset L_2(R)$ is compact.

Problem 24:

(i) Which of the following inclusions are correct?

(a) $f \in H^1(-1,1)$, where $f(x) = \text{sign } x$.

(b) $f \in H^1(-1,1)$, where $f(x) = |x|$.

(c) $f \in H^2(-1,1)$, where $f(x) = |x|$.

(ii) For which $\alpha \in R$ does the function $f(x) = |x|^{-\alpha} \sin |x|$ belong to $H^2(-1,1)$?

(iii) Let $Q = \{x \in R^n : |x| < 1\}$. For which $\alpha \in R$ and $k \geq 0$ does the inclusion $f \in H^k(Q)$ hold, where $f(x) = |x|^\alpha$?

Problem 25: For any function $f \in L_2(R^n)$ and $k \geq 0$ consider the function $g(\xi) = (1 + |\xi|^k)\hat{f}(\xi)$, where $\hat{f}(\xi)$ is the Fourier transform of the function $f(x)$.

(i) Prove that the following two statements are equivalent:

- $f \in H^k(R^n)$,
- $g \in L_2(R^n)$.

(ii) Prove that there exist constants $c_1, c_2 > 0$ such that the following inequalities hold for any $k \geq 0$ and $f \in H^k(R^n)$

$$c_1 \|g\|_{L_2(R^n)} \leq \|f\|_{H^k(R^n)} \leq c_2 \|g\|_{L_2(R^n)}.$$

Problem 26:

(i) Prove that for any real-valued function $f \in C^1[0, 2\pi]$ there exists a sequence $\{u_k \in R\}_{k=0}^{\infty}$ such that the series $\sum_{k=0}^{\infty} u_k \cos(kx/2)$ converges to the function f in $H^1(0, 2\pi)$.

Hint: Use that each of the systems $\{\sin(kx/2)\}_{k=1}^{\infty}$ and $\{\cos(kx/2)\}_{k=0}^{\infty}$ forms a basis for $L_2(0, 2\pi)$.

Bonus: Prove the same statement for an arbitrarily function $f \in H^1(0, 2\pi)$ (without using the notion of the trace of a function).

(ii) For a real-valued function $f \in H^1(0, 2\pi)$ prove the inequality

$$\int_0^{2\pi} f^2(x)dx \leq 4 \int_0^{2\pi} (f'(x))^2 dx + \frac{1}{2\pi} \left(\int_0^{2\pi} f(x)dx \right)^2 .$$

Hint: First prove it for $f \in C^1[0, 2\pi]$.

(iii) Find all real-valued functions $f \in C^1[0, 2\pi]$ for which the above inequality becomes an equality.