Homework assignment

Differentialgleichungen II

Pavel Gurevich, Sergey Tikhomirov

http://dynamics.mi.fu-berlin.de/lectures/12WS-Gurevich-PDE/

due date: 14:00, Tuesday, December 18, 2012

Problem 31:

(i) For a real-valued function $f \in L_2(0, \pi)$ consider the sequence

$$b_k = \frac{2}{\pi} \int_0^\pi f(x) \sin kx \, dx.$$

Prove that the function f is from the space $\dot{H}^1(0, \pi)$ (Sobolev space with zero trace) if and only if the series $\sum_{k \ge 1} k^2 b_k^2$ converges. Prove that if $f \in \dot{H}^1(0, \pi)$ then the following equality holds

$$\|f\|^2_{\dot{H}^1(0,\pi)} = \int_0^\pi (f^2 + f'^2) dx = \frac{\pi}{2} \sum_{k \ge 0} (k^2 + 1) b_k^2.$$

(ii) Let $x = (x_1, x_2) = (r \cos \varphi, r \sin \varphi)$ and a real-valued function

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k (a_k \cos k\varphi + b_k \sin k\varphi)$$

belong to $H^1(|x| < 1)$. Find the integral

$$\int_{|x|<1} (|\text{grad} f|^2 + |f|^2) dx$$

in terms of a_k, b_k.

Problem 32:

(i) Let Q be a bounded domain in R^n. Let $\rho \in C(\bar{Q})$ and $\rho(x) > \rho_0 > 0$. Prove that the expression

$$(f, g)_I = \int_Q \rho f \bar{g} \, dx \quad \text{for } f, g \in L_2(Q) \tag{1}$$

defines a scalar product in $L_2(Q)$ which is equivalent to the standard scalar product $(f, g) = \int_Q f \bar{g} \, dx$.

(ii) Let Q be a bounded domain in R^n. Let $\rho \in C(\bar{Q})$ and $\rho(x) > 0$ for $x \in Q \setminus \{x_0\}$ and $\rho(x_0) = 0$ for some $x_0 \in Q$. Prove that expression (1) defines a scalar product which is not equivalent to the standard scalar product.

Problem 33: Let Q be a bounded domain in R^n and D be a bounded domain in R^{n-1}. Let $\varphi : D \to R$ be a C^1 function. Denote $\Gamma_\delta = \{(x', \phi(x') + \delta),\ \text{with } x' \in D\}$. Assume that $\Gamma_\delta \subset Q$ for small enough δ. Let $f \in H^1(Q)$. Denote by $h_\delta \in L_2(\Gamma_\delta)$ the trace of f on Γ_δ. Let $g_\delta \in L_2(\Gamma_0)$ is defined by $g_\delta(x) = h_\delta(x - (0, \delta))$. Prove that

$$\lim_{\delta \to 0} \|g_\delta - g_0\|_{L_2(\Gamma_0)} = 0.$$

Problem 34: Let Q be a bounded domain and $c > 0$. Consider real-valued functions $q \in C(\bar{Q})$, $r \in C(\partial Q)$ satisfying inequality $q(x), r(y) > c$ for all $x \in Q$, $y \in \partial Q$. Let p be a matrix-valued function continuous in $\bar{Q}$ such that $p(x)$ is a symmetric positively defined $n \times n$ matrix satisfying $(p(x)\xi, \xi) \geq c|\xi|^2$ for all $x \in Q$ and $\xi \in R^n$. Prove that the norm in $H^1(Q)$ defined by the following scalar product

$$(f, g) = \int_Q (q f \bar{g} + (p \nabla f, \nabla \bar{g}))\ dx + \int_{\partial Q} r f \bar{g}\, dS, \quad f, g \in H^1(Q),$$

is equivalent to the classical norm in $H^1(Q)$.