Homework assignment

Differentialgleichungen II

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http://dynamics.mi.fu-berlin.de/lectures/12WS-Gurevich-PDE/

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Problem 35: Consider the elliptic problem

$$\begin{cases} -\sum_{i,j=1}^n (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^n a_i(x)u_{x_i} + a(x)u = f(x), \\ u|_{\partial Q} = 0, \end{cases} \tag{1}$$

where $\{a_{ij}(x)\}$ is positively definite, $a_{ij}, a_i \in C^1(\bar{Q})$, $a \in C(\bar{Q})$. Assume additionally that

$$a(x) - \frac{1}{2}\sum_{i=1}^n \frac{\partial a_i}{\partial x_i}(x) \geq 0, \quad x \in \bar{Q}.$$

Prove that problem (1) has a unique generalized solution for any $f \in L_2(Q)$.

Hint: use u as a test function.

Problem 36: Consider the equation

$$-\Delta u + \lambda u = f(x), \quad x \in R^n.$$

Find all $\lambda \in \mathbb{C}$ such that, for any $f \in L_2(R^n)$, there exists a unique solution $u \in H^2(R^n)$ of this equation.

Hint: Use the Fourier transform.

Problem 37: Let $Q \subset R^2$ be a bounded domain. Assume that

(i) $Q = \{|x| < 1\}$,

(ii) Q has C^1 boundary.

Let $\varphi \in C^1(\partial Q)$. Prove that there exists a function $u \in C^1(\bar{Q})$ such that $u|_{\partial Q} = \varphi$.

Hint: In (ii) use partition of unity.

Problem 38: Let $Q \subset R^n$ be a bounded domain with $\partial Q \in C^1$. Define

$$H^{1/2}(\partial Q) = \{\varphi \in L_2(\partial Q) : \text{ there exists } b \in H^1(Q) \text{ such that } b|_{\partial Q} = \varphi\}.$$

Remark: $H^{1/2}(\partial Q) \neq L_2(\partial Q)$.

Consider the problem

$$\begin{cases} -\Delta u = f(x), & x \in Q, \\ u|_{\partial Q} = \varphi, \end{cases} \tag{2}$$

where $f \in L_2(Q)$, $\varphi \in H^{1/2}(\partial Q)$.

Let $b \in H^1(Q)$ be such that $b|_{\partial Q} = \varphi$ (in sense of traces). We say that a function $u \in H^1(Q)$ is a generalized solution of (2) if for any $v \in \mathring{H}^1(Q)$ the following holds

$$\int_Q \nabla w \nabla \bar{v}\, dx = \int_Q f \bar{v}\, dx - \int_Q \nabla b \nabla \bar{v}\, dx, \tag{3}$$

where $w = u - b$.

Note that the notion of a solution of (2) depends on a choice of b.

(i) Prove that if the functions $u, b \in C^2(\bar{Q})$, u satisfies (2) and $v \in \dot{C}^1(Q)$, then (3) holds.

(ii) Prove that for a fixed function b there exists a unique generalized solution of (2).

(iii) Prove that, in fact, the general solution depends on f and on φ, but does not depend on a particular choice of b. To do so consider two different functions $b_1, b_2 \in H^1(Q)$ such that $b_1|_{\partial Q} = b_2|_{\partial Q} = \varphi$. Let $u_1, u_2 \in H^1(Q)$ be generalized solutions of (2) with respect to the choices b_1, b_2. Prove that $u_1 = u_2$.