Homework assignment

Differentialgleichungen II

Pavel Gurevich, Sergey Tikhomirov

http://dynamics.mi.fu-berlin.de/lectures/12WS-Gurevich-PDE/

due date: 14:00, Tuesday, January 15, 2012

Problem 39: Let $Q \subset R^n$ be a bounded domain with a smooth boundary. Consider a mixed boundary-value problem.

$$\begin{cases} -\Delta u = f(x), & x \in Q, \\ \left(\frac{\partial u}{\partial \nu} + \rho(x)u\right)\big|_{\partial Q} = g, \end{cases} \tag{1}$$

where $f \in L_2(Q)$, $g \in L_2(\partial Q)$, $\rho \in C^\infty(\partial Q)$, $\rho(x) \geq 0$ and there exists $x_0 \in Q$ such that $\rho(x_0) \neq 0$.

We say that a function $u \in H^1(Q)$ is a *generalized solution* of (1) if for all $v \in H^1(Q)$ the following integral equality holds:

$$\int_Q (\nabla u \nabla v)\, dx + \int_{\partial Q} \rho uv\, dS = \int_Q fv\, dx + \int_{\partial Q} gv\, dS.$$

(i) Prove that if $u \in C^2(\bar{Q})$ is a classical solution of (1) then u is a generalized solution.

(ii) Prove that for any $f \in L_2(Q)$ and $g \in L_2(\partial Q)$ a generalized solution exists and is unique.

Remark: Note that for functions $u \in H^1(Q)$ the expression $\frac{\partial u}{\partial \nu}$ in general is not defined.

Problem 40: Find a function v_0 which realizes the infimum of the functional

$$\int_0^1 (v'^2 + v^2)\, dx + 2\int_0^1 v\, dx$$

in the space $\mathring{H}^1(0,1)$.

Problem 41: Prove that for any $v \in C^1([0,1])$ the following inequality holds

$$\int_0^1 (v'^2 + 2xv)dx + v^2(0) + v^2(1) \geq -\frac{41}{270}.$$

Is there a function $v \in C^1([0,1])$ for which the left and right-hand sides are equal?

Problem 42: Let $Q \subset R^n$ be a bounded domain with a smooth boundary. For a function $f \in L_2(Q)$ and $g \in H^{1/2}(\partial Q)$ consider the set $A := \{v \in H^1(Q) : \ v|_{\partial Q} = g\}$ and a functional $E : A \to R$ defined by the formula

$$E(v) = \int_Q |\nabla v|^2 \, dx - 2 \int_Q f v \, dx.$$

Prove that there exists the unique function v, which realizes the infimum of E on set A and it is the generalized solution of the problem

$$\begin{cases} -\Delta v = f, \\ v|_{\partial Q} = g. \end{cases}$$