Homework assignment

Differentialgleichungen II

Pavel Gurevich, Sergey Tikhomirov

http://dynamics.mi.fu-berlin.de/lectures/12WS-Gurevich-PDE/

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Problem 43:

(i) Consider the operator $L : L_2(R^n) \to L_2(R^n)$, $Dom(L) = H^2(R^n)$, $Lu = -\Delta u$.

(a) Prove or disprove that L is bounded.

(b) Prove or disprove that L is closed. (Operator L is closed if its graph is closed, i.e. for any $u_n \in Dom(L)$ and $u, f \in L_2$, satisfying $u_n \to u$ in L_2 and $Lu_n \to f$ in L_2, holds the following $u \in Dom(L)$ and $Lu = f$.)

(c) Describe the spectrum of L. (Recall that spectrum of an operator is the set of $\lambda \in \mathbb{C}$, such that the operator $(L - \lambda \cdot \mathrm{Id})$ does not have a bounded inverse).

(ii) Let $Q \subset R^n$ be a bounded domain with the C^1 boundary. Consider the operator $P : L_2(Q) \to L_2(Q)$, $Dom(P) = H^2(Q)$, $Pu = -\Delta u$

(a) Prove or disprove that P is bounded.

(b) Prove or disprove that P is closed.

(c) Prove that $\lambda \in \mathbb{C}$ is in the spectrum of P if and only if there exists a nontrivial generalized solution of the following problem

$$\begin{cases} -\Delta u = \lambda u, & x \in Q, \\ u|_{\partial Q} = 0. \end{cases}$$

Problem 44: Let $Q \subset \mathbb{R}^2 \cong \mathbb{C}$ be a simply connected bounded domain. Prove that a function $u \in C^2(Q)$ satisfies $\Delta u = 0$ in Q if and only if there exists a complex-analytic function $v : Q \to \mathbb{C}$ such that $u(x, y) = \mathrm{Re}\ v(x + iy)$.

Hint: Use the Cauchy–Riemann equations.

Problem 45: Let $Q \subset R^n$ be a bounded domain. Let u be a harmonic ($\Delta u = 0$) function in Q. Prove that $u \in C^\infty(Q)$.

Problem 46: Let $Q \subset R^n$ be a domain. A function $G : \bar{Q} \times Q \to R$ is called a Green function if it satisfies the following

- For any fixed $y \in Q$,
$$G(x, y) = U(x - y) + g(x, y),$$
where U is the fundamental solution of the Laplace operator and $g(x, y)$ is harmonic in Q and continuous in $\bar{Q}$ with respect to variable x.

- For any $y \in Q$,
$$G(x, y)|_{x \in \partial Q} = 0.$$

(i) Prove that if $u \in C^2(Q) \cap C^1(\bar{Q})$ satisfies
$$\begin{cases} \Delta u = f(x), \\ u|_{\partial Q} = \varphi(x) \end{cases}$$
then the following holds for any $x \in Q$
$$u(x) = \int_Q G(x, y) f(y) dy + \int_{\partial Q} \frac{\partial G(x, y)}{\partial \nu_y} \varphi(y) dS_y.$$

(ii) Find the Green function for the following domains.
(a) $Q \subset R^3$, $Q = \{x \in \mathbb{R}^3 : \ x_1 > 0\}$.
(b) $Q \subset R^3$, $Q = \{x \in \mathbb{R}^3 : \ x_1, x_2 > 0\}$.
(c) $Q \subset R^3$, $Q = \{|x| < 1\}$.

Hint: Try to construct $g(x, y)$ as a linear combination of the fundamental solutions $U(x - y^{(i)})$ for suitable $y^{(i)} \in R^n$.

Remark. Physical interpretation of the fundamental solution $U(x-y)$ and the Green function $G(x, y)$ in $\mathbb{R}^3$.
$U(x - y)$ is the potential of the electric field at the point $x \in \mathbb{R}^3$ created by a charge $-1/(4\pi)$ located at the point $y \in \mathbb{R}^3$.
$G(x, y)$ is the potential of the electric field at the point $x \in Q$ created by a charge $-1/(4\pi)$ located at the point $y \in Q$, provided that the surface ∂Q is grounded (has the zero potential).

(iii) Let $Q = \{x \in \mathbb{R}^3 : |x| < R\}$. Prove that the problem
$$\begin{cases} \Delta u = 0, \\ u|_{\partial Q} = \varphi, \end{cases}$$
where $\varphi \in C(\partial Q)$, have a classical solution defined by the formula
$$u(x) = \frac{1}{4\pi R} \int_{\partial Q} \frac{R^2 - |x|^2}{|x - y|^3} \varphi(y) \, dS.$$