Homework assignment

Differentialgleichungen II

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http://dynamics.mi.fu-berlin.de/lectures/12WS-Gurevich-PDE/

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Problem 47: Let

$$\mathcal{E}(x,t) = \frac{1}{(2\sqrt{\pi t})^n} e^{-\frac{|x|^2}{4t}} \tag{1}$$

be the Poisson kernel. Prove the following statements.

(i) For any $t > 0$ the following holds

$$\int_{R^n} \mathcal{E}(x,t)dx = 1.$$

(ii) For any $\delta > 0$ the following holds

$$\lim_{t\to 0} \int_{R^n \setminus B_\delta(0)} \mathcal{E}(x,t)dx = 0.$$

Problem 48: Let $u \in C^\infty(R^n \times [0,T])$ for some $T > 0$. Assume there exists $L > 0$ such that $u(x,t) = 0$ for $|x| > L$, $t \in [0,T]$. Denote

$$f(x,t) = u_t - \Delta u, \quad \varphi(x) = u(x,0).$$

Prove the Poisson formula using Fourier transform

$$u(x,t) = \int_0^t \int_{R^n} \mathcal{E}(x-y,t-s) f(y,s)\, dy\, ds + \int_{R^n} \mathcal{E}(x-y,t)\varphi(y)\, dy.$$

Problem 49: For each of the following problems find the solution

(i) $u_t = u_{xx} + 3t^2$; $u|_{t=0} = \sin x$, for $x \in R^1$, $t \geq 0$;

(ii) $8u_t = \Delta u + 1$; $u|_{t=0} = e^{-(x-y)^2}$, for $(x,y) \in R^2$, $t \geq 0$;

(iii) $u_t = 3\Delta u + e^t$; $u|_{t=0} = \sin(x-y-z)$, for $(x,y,z) \in R^3$, $t \geq 0$.

Problem 50: Consider a function $\varphi \in C(R^n)$. Let $a = 1$.

(i) Assume that for some $\delta > 0$ there exists $M_\delta > 0$ such that $|\varphi(x)| \le M_\delta e^{\delta |x|^2}$. Prove that for $t \in \left(0, \frac{1}{4a^2\delta}\right)$ and $x \in R^n$ the function

$$u(x,t) = \int_{R^n} \mathcal{E}(x-y,t)\varphi(y)\,dy \tag{2}$$

is well-defined and belongs to the class C^∞. Prove that $u(x,t)$ solves the Cauchy problem

$$u_t = a^2 \Delta u, \quad u|_{t=0} = \varphi(x), \quad t \in \left(0, \frac{1}{4a^2\delta}\right).$$

(ii) Assume that the conditions of section (i) holds for any $\delta > 0$. Prove that the function $u(x,t)$ defined by the same formula belongs to the class $C^\infty(R^n \times R^+)$ and solves the Cauchy problem

$$u_t = a^2 \Delta u, \quad u|_{t=0} = \varphi(x), \quad t \ge 0. \tag{3}$$

(iii) Assume that there exists $L > 0$ such that $\varphi(x) = 0$ for $|x| > L$. Let $u(x,t)$ be the solution of problem (3). Prove that for any $T > 0$, $\delta \in (0, \frac{1}{4a^2T})$ there exists $M > 0$ such that

$$u(x,t) \le M e^{-\delta |x|^2}, \quad x \in R^n, \quad t \in (0,T).$$

(iv) **Bonus:** Assume that $a \ne 1$. Find modification of the definition of $\mathcal{E}(x,y)$ such that formula (2) gives a solution of (3).