Homework assignment

Differentialgleichungen II

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http://dynamics.mi.fu-berlin.de/lectures/12WS-Gurevich-PDE/

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Let D be a bounded domain in R^n with C^∞ boundary. For any $T > 0$ denote $Q_T = D \times (0,T)$, $\Gamma_T = \partial D \times (0,T)$, $D_\tau = \{x \in D, t = \tau\}$, $\tau \in R$.

Problem 51: Fix $T > 0$ and consider a hyperbolic initial boundary-value problem in Q_T

$$u_{tt} - \Delta u = f(x,t), \quad (x,t) \in Q_T, \tag{1}$$

$$u|_{t=0} = \varphi(x), \quad x \in D, \tag{2}$$

$$u_t|_{t=0} = \psi(x), \quad x \in D, \tag{3}$$

$$u|_{\Gamma_T} = 0, \tag{4}$$

where f, φ, ψ are L_2 functions in the corresponding spaces.

(i) Prove that if a generalized solution u is an $H^2(Q_T)$ function then equation (1) holds almost everywhere and (3) holds in the sense of traces.

(ii) Prove that if a function $u \in C^2(Q_T) \cap C^1(Q_T \cup \bar{D}_0 \cup \Gamma_T)$ satisfies equations (1)-(4) in the classical sense, then for any $T' \in (0,T)$ the function u is a generalized solution of problem (1-4) in $Q_{T'}$. Prove or disprove that the function u is necessarily a generalized solution in Q_T.

(iii) Assume $f(x,t) = 0$. Let u be a generalized solution of (1)-(4). Prove that for any $\tau \in (0,T)$ the following equality holds:

$$\int_{D_\tau} (u_t^2 + |\nabla u|^2) dx = \int_{D_0} (\psi^2 + |\nabla \varphi|^2) dx.$$

Problem 52: **(200 points)**

Fix $T > 0$ and consider a parabolic initial boundary-value problem in Q_T

$$u_t - \Delta u = f(x,t), \quad (x,t) \in Q_T, \tag{5}$$

$$u|_{t=0} = \varphi(x), \quad x \in D, \tag{6}$$

$$u|_{\Gamma_T} = 0, \tag{7}$$

where $f \in L^2(Q_T)$. In the lectures it was proved by the Fourier method that if $\varphi \in L_2(D)$ then there exists a generalized solution $u \in H^{1,0}(Q_T)$ of problem (5)-(7).

Carefully follow the same steps as in that proof and show that if $\varphi \in \mathring{H}^1(D)$ then the generalized solution u belongs to $H^{2,1}(Q_T)$.

Remark:

$$\|u\|^2_{H^{2,1}(Q_T)} = \|u\|^2_{L_2(Q_T)} + \|u_t\|^2_{L_2(Q_T)} + \sum_{i=1}^{n} \|u_{x_i}\|^2_{L_2(Q_T)} + \sum_{i,j=1}^{n} \|u_{x_i x_j}\|^2_{L_2(Q_T)}.$$

Problem 53: Solve, using the Fourier method.

(i) $u_{tt} - u_{xx} + 2u_t = 4x + 8e^t \cos x,\ x \in (0, \pi/2)$;
$u_x|_{x=0} = 2t,\ u_x|_{x=\pi/2} = \pi t,\ u|_{t=0} = \cos x,\ u_t|_{t=0} = 2x.$

(ii)

$$u_t = \Delta u + 1 \quad (0 < x < \pi,\ 0 < y < 2\pi),$$
$$u_x|_{x=0} = u_x|_{x=\pi} = 0, \qquad u_y|_{y=0} = u_y|_{y=2\pi} = 0,$$
$$u|_{x=0} = 2\cos x \cos 2y + 1.$$

Problem 54: Prove or disprove compactness of the following embeddings

(i) $H^2(R) \subset H^1(R)$;

(ii) $H^2(Q) \subset H^1(Q)$, where $Q \subset R^n$ is a bounded domain with C^∞ boundary.

Problem 55:

(i) Let $u \in H^1(0,l)$. Prove that

$$\int_0^l |u'|^2 dx \geq \frac{|u(l) - u(0)|^2}{l}.$$

(ii) Prove that for any $f \in \mathring{H}^1(a,b)$ the following holds:

$$\int_a^b f^2 dx \leq \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'|^2 dx.$$