Theoretical questions for the exam on Differential Equations II. Winter semester 2012/2013

Nonlinear first-order PDE. Characteristics

1. Basic notions: derivatives and multi-index, support of a function, smoothness of a boundary of domain, integration by parts, classification of PDE w.r.t. (non)linearity.
2. General setting of first-order PDE. Example: physical derivation of the conservation law.
3. Characteristic equations.
   Theorem: if u(x) is a solution of PDE, then the characteristic equations "hold".
4. Characteristic equations in particular cases:
   a. linear PDE,
   b. quasilinear PDE.
5. Initial data for characteristic:
   a. straightening the boundary (reduction to PDE in a neighborhood of flat boundary),
   b. admissible initial data at a point x_0 on the (flat) boundary,
   c. noncharacteristic initial data at a point x_0 on the (flat) boundary,
   d. **Lemma** about initial data in a neighborhood of x_0,
   e. **Lemma (local invertibility)**: "trajectories of characteristic equations fill up a neighborhood of x_0 and do not intersect there.
6. **Theorem** about existence and uniqueness of local solutions.

Linear second-order PDE. I

Cauchy problem

7. Setting of the Cauchy problem. Straightening the boundary (reduction to PDE in a neighborhood of flat boundary).
8. (Non)characteristic surface:
   a. definition,
   b. necessary conditions for expressing a solution and its derivatives on the boundary via the data of the problem,
   c. **Theorem**: If $u(x)$ is a solution in a neighborhood of a characteristic point, then it satisfies an additional relation.
9. Kovalevskaya's theorem on existence and uniqueness of local solutions:
   a. analytic complex-valued functions of several variables,
   b. **Kovalevskaya's theorem** (unfortunately without proof).

Classification. Setting of problems of mathematical physics

10. Classification of linear second-order PDE:
    a. elliptic PDE. Poisson (Laplace) equation. Any surface is noncharacteristic,
    b. hyperbolic PDE. Wave equation. Characteristic surfaces (cones),
    c. ultrahyperbolic PDE,
    d. parabolic PDE. Heat (diffusion) equation. Characteristic surfaces (hyperplanes).
11. Invariance of classification under nondegenerate change of variables. Reduction to the canonical form.
12. Setting of problems of mathematical physics:
    a. boundary-value problems for elliptic equations (Poisson equation). Dirichlet, Neumann, and Robin boundary conditions. Physical meaning.

b. Cauchy problem and initial boundary-value problem for parabolic equations (heat equation). Physical meaning.
c. Cauchy problem and initial boundary-value problem for hyperbolic equations (wave equation). Physical meaning.

Functional spaces

Lebesgue space $L_2(Q)$ for a bounded domain Q

13. Mollifiers. Construction of a cut-off function.
14. Definition of the Lebesgue space $L_2(Q)$ for a bounded domain Q. Scalar product and norm.
    Theorem: $C(\bar{Q})$ is dense in $L_2(Q)$.
    Corollary: $L_2(Q)$ is separable.
15. **Theorem**: $L_2(Q)$ functions are square-integrably continuous (mean-square continuous).
16. **Theorem** about approximation by mollifiers.
    Corollary: $\dot{C}^\infty(Q)$ is dense in $L_2(Q)$.

Fourier transform

17. Fourier transform for functions from $L_1(R^n)$. Basic properties.
18. Schwartz space $S(R^n)$ of rapidly decreasing functions.
    Lemma: Fourier transform is a linear continuous map from $S(R^n)$ to $S(R^n)$.
    Lemma: $S(R^n)$ is dense in $L_1(R^n)$ and $L_2(R^n)$.
19. **Theorem** about the Fourier transform being a linear continuous one-to-one map from $S(R^n)$ *onto* $S(R^n)$. The inverse Fourier transform.
20. Parseval identity for functions from the Schwartz space $S(R^n)$.
    Plancherel theorem: extension of the Fourier transform as a linear continuous one-to-one map from $L_2(R^n)$ onto $L_2(R^n)$.

Generalized derivatives

21. Definition of a generalized derivative for $L_{2,\mathrm{loc}}(Q)$ functions. Relation to a classical derivative. Uniqueness.
    Lemma about approximation of a generalized derivative by derivatives of its mollifiers.
    Corollary: a function is constant if all first generalized derivatives are zero.
22. Theorem: criterion for existence of generalized derivatives (in terms of mollifiers).

Sobolev spaces

23. Definition. Basic properties.
    Lemma: smooth functions on a parallelepiped are dense in Sobolev spaces.
24. **Lemma**: continuation (extension) of functions defined in a parallelepiped.
25. **Lemma**: existence of partition of unity.
26. **Theorem**: continuation (extension) of functions defined in a bounded domain.
    Theorem: smooth functions in a bounded domain are dense in Sobolev spaces.
27. **Theorem**: separability of Sobolev spaces.
28. **Rellich-Kondrachov (Rellich-Gårding) theorem**: $H^1(Q)$ is compactly embedded into $L_2(Q)$.
29. Trace of a function.
30. Integration by parts for $H^1(Q)$ functions.
31. Sobolev space $\dot{H}^1(Q)$.

a. Definition.
b. **Theorem**: $\dot{H}^1(Q)$ is the set of $H^1(Q)$ functions with zero trace on the boundary (without proof).
c. **Theorem** about the equivalent scalar product.
d. **General theorem** about equivalent scalar products in $H^1(Q)$.

Linear second-order PDE. II

Boundary-value problems for elliptic equations

32. Dirichlet problem for the Poisson equation:
    a. setting of problem; notion of a generalized solution,
    b. **Lax-Milgram theorem** about existence and uniqueness of a generalized solution.
33. Dirichlet problem for general elliptic equations:
    a. setting of problem; notion of a generalized solution,
    b. adjoint problem,
    c. Fredholm solvability (three theorems).
34. Eigenvalues and eigenfunctions of the Dirichlet problem for the Laplace equation:
    a. definition of eigenvalues and generalized eigenfunctions,
    b. reduction to a compact operator,
    c. **Theorem** about discreteness of eigenvalues and orthogonality of eigenfunctions,
    d. **Theorem** about expansion of $\dot{H}^1(Q)$ functions into Fourier series w.r.t. eigenfunctions.
35. Neumann problem for the Poisson equation:
    a. setting of problem; notion of a generalized solution,
    b. **Theorem** about Fredholm solvability.
36. Eigenvalues and eigenfunctions of the Neumann problem for the Laplace equation (see Q. 34).
37. Variational and boundary-value problems. Ritz method:
    a. abstract "energy" functional; minimizing sequence; minimizer,
    b. **Theorem** about existence and uniqueness of a minimizer,
    c. Ritz method: minimizing Ritz sequence,
    d. **Theorem** about the connection with the Dirichlet problem.
38. Generalized derivatives and finite differences:
    a. **Theorem** compactly supported functions,
    b. **Theorem** for symmetric domains.
39. Regularity of generalized solutions for the Dirichlet problem:
    a. **Theorem** about local regularity,
    b. **Corollary**: generalized solution satisfies the Poisson equation a.e. in Q,
    c. **Theorem** about global regularity (up to the boundary of Q).
40. Connection between generalized and classical solutions:
    a. **Sobolev theorem** about continuous embedding of Sobolev spaces into spaces of continuous functions,
    b. Existence of classical solutions of the Dirichlet problem for the Poisson equation,
    c. Regularity of generalized eigenfunctions.
41. Classical solutions of the Dirichlet problem for the Poisson equation via the fundamental solution and integral representation:
    a. definition of a classical solution,
    b. the fundamental solution,
    c. **Theorem** about the integral representation in a bounded domain,
42. Some properties of harmonic functions:
    a. Mean-value theorems,
    b. **Theorem** about maximum principle,

c. **Theorem** about uniqueness of classical solutions of the Dirichlet problem for the Poisson equation,
d. **Theorem** about existence of classical solutions in a bounded domain (without proof).

Initial boundary-value problem and Cauchy problem for parabolic (heat) equation

43. Anisotropic Sobolev space $H^{1,0}(Q_T)$. Its properties.
44. Initial boundary-value problem for the heat equation:
    a. setting of the problem, definition of a generalized solution,
    b. **Theorem** about uniqueness of a generalized solution,
    c. **Theorem** about existence of a generalized solution (via the Fourier method).
    d. **Theorem** about regularity of a generalized solution (without proof).
45. Cauchy problem:
    a. setting of the problem, definition of a classical solution in a strip,
    b. the fundamental solution (heat kernel) and its properties,
    c. **Theorem** about the integral representation (Poisson formula),
    d. **Theorem** about uniqueness of classical solutions,
    e. **Theorem** about existence of classical solutions.

Initial boundary-value problem and Cauchy problem for hyperbolic (wave) equation

46. Initial boundary-value problem for the wave equation:
    a. setting of the problem, definition of a generalized solution,
    b. **Theorem** about uniqueness of a generalized solution,
    c. **Theorem** about existence of a generalized solution (via the Fourier method),
    d. **Theorem** about existence of a generalized solution (via the Galerkin method).
47. Cauchy problem:
    e. setting of the problem, definition of a classical solution in a strip,
    f. **Theorem** about the integral representation for $x \in \mathrm{R}^3$ (Kirchhoff formula),
    g. **Theorem** about uniqueness of classical solutions,
    h. **Theorem** about existence of classical solutions (without proof),
    i. Poisson and d'Alembert formulas
    j. Regions of dependence of solutions on the data of the Cauchy problem