Homework assignment

Differentialgleichungen III

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http://dynamics.mi.fu-berlin.de/lectures/13SS-Gurevich-Dynamics/

Tutorial discussion date: Friday, April 19, 2013, at 10:00

Problem 1: A semigroup of linear bounded operators $T(t)$, $t \geq 0$, in a Banach space X is said to be a *strongly continuous* or C_0 *semigroup* if

$$T(t)u \to u \quad \text{as } t \to 0 \quad \text{for all } u \in X.$$

It is known that

(i) there is $\omega \in \mathbb{R}$ such that
$$\|T(t)\| \leq e^{\omega t} \quad \forall t \geq 0,$$

(ii) the generator of a strongly continuous semigroup is a linear closed densely defined operator.

Questions:

(i) **(Continuity)** Let $T(t)$ be a strongly continuous semigroup. Prove that

$$T(t)u \to T(t_0)u \quad \text{as } t \to t_0 \quad \text{for all } t_0 > 0,\ u \in X.$$

(ii) **(Commutation and differentiation)** Let $L : X \to X$ be a generator of a strongly continuous semigroup $T(t)$. Let $u \in D(L)$. Show that $T(t)u \in D(L)$ and

$$LT(t)u = T(t)Lu = \frac{d}{dt}T(t)u \quad \forall t \geq 0.$$

(iii) **(Uniqueness)** Let $L : X \to X$ be a generator of strongly continuous semigroups $T(t)$ and $S(t)$. Prove that
$$T(t) = S(t) \quad \forall t \geq 0.$$

Hint: differentiate $G(s) = T(s)S(t-s)$.

(iv) **(Shift)** Let $L : X \to X$ be a generator of a strongly continuous semigroup $T(t)$. Show that $L + aI$, $a \in \mathbb{C}$, generates the strongly continuous semigroup $e^{at}T(t)$.

Problem 2: Let $Q \subset \mathbb{R}^n$ be a bounded domain with $\partial Q \in C^2$. Consider the operator

$$A : L_2(Q) \to L_2(Q), \quad D(A) = \mathring{H}^1(Q) \cap H^2(Q),$$
$$Au = -\Delta u, \quad u \in D(A).$$

Prove that A is sectorial.

Problem 3: Consider the operator A from Problem 2. Write explicit formulas for Au ($u \in D(A)$) and $e^{-At}v$ ($v \in L_2(Q)$) via the Fourier representation with respect to eigenfunctions of A.

Hint: use the integral representation of e^{-At} from the lecture.