Homework assignment

Differentialgleichungen III

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http://dynamics.mi.fu-berlin.de/lectures/13SS-Gurevich-Dynamics/

Tutorial discussion date: Friday, April 26, 2013, at 10:00am

Problem 1:

(i) Let A be a sectorial operator. Prove that

$$A\int_\Gamma e^{\lambda t}(\lambda + A)^{-1}\,d\lambda = \int_\Gamma e^{\lambda t} A(\lambda + A)^{-1}\,d\lambda \quad \forall t > 0,$$

where Γ is a contour in the resolvent set of $-A$ and the integral converges in the uniform operator topology (see Theorem 1 from Lecture 1).

(ii) Let A be a sectorial operator. Prove equality (3) in Theorem 1 from Lecture 1:

$$\frac{d}{dt}e^{-At} = -Ae^{-At} \quad \forall t > 0,$$

where the derivative and the equality are understood in the uniform operator topology.

Problem 2:

(i) Let A be a sectorial operator and B a bounded linear operator. Prove that $A+B$ is sectorial.

(ii) **Bonus:** Generalize assertion (i) to the situation where B is unbounded and

(a) $D(A) \subset D(B)$,

(b) for any $\varepsilon > 0$ there is $K(\varepsilon) > 0$ such that

$$\|Bu\| \le \varepsilon\|Au\| + K(\varepsilon)\|u\| \quad \forall u \in D(A).$$

Problem 3:

(i) Prove that the operator

$$B : L_2(0,1) \to L_2(0,1),$$
$$D(B) = \{z \in H^2(0,1) : z'(0) = 0,\ z'(1) + z(1) = 0\},$$
$$Bz(x) = -z''(x)$$

is sectorial.

(ii) Consider the operator corresponding to the one-dimensional elliptic equation with *nonhomogeneous* boundary conditions

$$A : L_2(0,1) \times \mathbb{R}^2 \to L_2(0,1) \times R^2,$$
$$D(A) = \{(u,v,w) \in H^2(0,1) \times \mathbb{R}^2 : u'(0) = v,\ u'(1) + u(1) = w\},$$
$$Au(x) = -u''(x).$$

Prove that A is sectorial.

Hint:

(a) If $(\lambda - A)(u,v,w) = (f,g,h) \in L^2 \times \mathbb{R} \times \mathbb{R}$, $\lambda \neq 0$, then $v = g\lambda^{-1}, w = h\lambda^{-1}$, and $u'' + \lambda u = f$ with the corresponding boundary conditions.

(b) Introduce the function $z(x) = u(x) - xv + 2v - w$. Show that z is in the domain of the operator B.

Problem 4:

(i) Let $A : X \to X$ be a sectorial operator with $\operatorname{Re}\sigma(A) > \delta > 0$. Prove that A^{-1} defined via the Laplace transform of e^{-At} coincides with the inverse of A

(ii) Let $A : L_2(Q) \to L_2(Q)$ be the "minus Laplace" operator defined in question 2 in problem sheet 1. Show that, for each $\alpha > 0$, $A^{-\alpha} : L_2(Q) \to L_2(Q)$ is a bounded operator given by

$$A^{-\alpha}u(x) = \sum_{k=1}^{\infty} \lambda_k^{-\alpha} u_k e_k(x),$$

where λ_k and $e_k(x)$ are eigenvalues and eigenfunctions of A and $u_k = (u, e_k)_{L_2(Q)}$.