Homework assignment

Differentialgleichungen III
Problem Sheet 3

Pavel Gurevich, Eyal Ron
http://dynamics.mi.fu-berlin.de/lectures/13SS-Gurevich-Dynamics/

Tutorial discussion date: Friday, May 3, 2013, at 10:00

Problem 1: Let $\alpha \in (0,1)$. Use the Cauchy integral to prove the formula from the lecture

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda + A)^{-1} d\lambda$$

when A is a positive real number.

Problem 2: Let $A : L^2(Q) \to L^2(Q)$ be the "minus Laplace" operator defined in question 2 in problem sheet 1. Find the domain of A^α, $\alpha > 0$. Represent $A^\alpha u(x)$, $u \in D(A^\alpha)$, via the Fourier series with respect to the eigenfunctions of A.

Problem 3: Let A be a sectorial operator such that $Re\,\sigma(A) > \delta > 0$. Prove each of the following properties for fractional powers of A.

(i) If $\alpha \geq \beta > 0$, then $D(A^\alpha) \subset D(A^\beta)$.

(ii) If $\alpha > 0$, A^α is closed and densely defined. **Hint:** Use the fact that A^n is densely defined for every natural n (see also **Bonus**).

(iii) $A^\alpha A^\beta = A^{\alpha+\beta}$ on $D(A^\gamma)$, where $\gamma = \max(\alpha, \beta, \alpha + \beta)$.

(iv) $A^\alpha e^{-At} = e^{-At} A^\alpha$ on $D(A^\alpha)$, $t > 0$.

Bonus: Prove that $\bigcap\limits_{n=1}^{\infty} D(A^n)$ is dense in X.

Problem 4:

(i) Let A be a sectorial operator such that $Re\,\sigma(A) > \delta > 0$. Let $\alpha \in (0,1)$. Show that if $u \in D(A)$ then $||A^\alpha u|| \leq C||Au||^\alpha ||u||^{1-\alpha}$.

Hint: 1. Estimate $\|A^{-\beta}v\|$, splitting the integral into two: $\int_0^\varepsilon + \int_\varepsilon^\infty$.

2. Minimize over $\varepsilon > 0$.

3. Set $\alpha = 1 - \beta$ and $v = Au$.

(ii) Conclude from (i) that $||A^\alpha u|| \leq \varepsilon||Au|| + C'\varepsilon^{-\alpha/(1-\alpha)}||u||$ for all $\varepsilon > 0$. (Here C, C' are constants independent of u. Do they depend on α?)