Homework assignment

Differentialgleichungen III
Problem Sheet 4

Pavel Gurevich, Eyal Ron

http://dynamics.mi.fu-berlin.de/lectures/13SS-Gurevich-Dynamics/

Tutorial discussion date: Friday, May 17, 2013, at 10:00am

Problem 1:

(i) Let A, B be sectorial operators such that $\mathrm{Re}\,\sigma(A), \mathrm{Re}\,\sigma(B) > 0$. Show that if $(A - B)A^{-\beta}$ is bounded for $\beta \in [0, 1)$, then $A^\alpha B^{-\alpha}$ and $B^\alpha A^{-\alpha}$ are bounded for $\alpha = 0, 1$.

(ii) Let A be a sectorial operator with $\mathrm{Re}\,\sigma(A) > 0$. Show that for all $u \in X$, $(I + \varepsilon A)^{-1}v \to v$ in X as $\varepsilon \to 0$.

(iii) Let A be a sectorial operator with $\mathrm{Re}\,\sigma(A) > 0$. Show that $(I + A)^{-1}A^{-\beta} = A^{-\beta}(I + A)^{-1}$ for $\beta > 0$.

(iv) Let A be a closed operator. Show that if the resolvent is compact at one point, then it is compact everywhere.

Problem 2: Let A be a sectorial operator in X with $\mathrm{Re}\,\sigma(A) > 0$. Let X_1, X_2 be the invariant spaces defined in the lecture, and A_1, A_2 the restrictions of A on X_1, X_2. Prove the following facts:

(i) $D(A_1^\alpha) = X_1$, and $A_1^\alpha : X_1 \to X_1$ is bounded $\forall \alpha > 0$.

(ii) $D(A_2^\alpha) = D(A^\alpha) \cap X_2$, $\forall \alpha > 0$.

(iii) $e^{-At}X_j \subset X_j$, $t \geq 0$.

(iv) $e^{-At}|_{X_j} = e^{-A_j t}$.

Problem 3: Consider the problem

$$\begin{aligned} &u_t = u_{xx}, \qquad x \in (0,1), t > 0, \\ &u_x(0,t) = v(t), \\ &u_x(1,t) + u(1,t) = w(t), \end{aligned} \tag{1}$$

where v and w satisfy the differential equations

$$\begin{aligned} &\dot{v} = \alpha v + \beta w, \\ &\dot{w} = \gamma v + \delta w + \int_0^1 u(x,t)dx, \end{aligned} \tag{2}$$

$\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Write (1), (2) in the form

$$\frac{d}{dt}(u, v, w) = A(u, v, w).$$

Define $D(A)$ properly. Prove that A is sectorial in $L^2(0,1) \times \mathbb{C}^2$.
Hint: See problems 2 and 3(ii) in problem sheet 2.