Homework assignment

Differentialgleichungen III
Problem Sheet 5

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http://dynamics.mi.fu-berlin.de/lectures/13SS-Gurevich-Dynamics/

Tutorial discussion date: Friday, May 24, 2013, at 10:00am

Problem 1: Let Q be a bounded domain in $\mathbb{R}^n$, $\partial Q \in C^2$. Consider the heat equation:

$$\begin{aligned} &u_t = \Delta u + F(t,x), \qquad x \in Q, t > 0, \\ &u|_{\partial Q} = 0. \end{aligned} \tag{1}$$

Define $f(t) : L_2(Q) \to L_2(Q)$ as $f(t)(x) = F(t,x)$. Assume that $f(t)$ is locally Hölder continuous and integrable near $t = 0$. Show that if

$$||f(t) - f_0||_{L_2(Q)} \to 0 \quad \text{as} \quad t \to \infty$$

for some $f_0 \in L_2(Q)$, then $u(t,x)$, the solution of (1), satisfies

$$||u(t,\cdot) - u_0||_{L_2(Q)} \to 0 \quad \text{as} \quad t \to \infty,$$

where $u_0(x)$ is a solution of

$$\begin{aligned} &-\Delta u_0 = f_0(x), \qquad x \in Q, \\ &u_0|_{\partial Q} = 0. \end{aligned}$$

Hints:

(i) Formulate the heat equation as $u_t + Au = f(t)$ in $X = L_2(\mathbb{R})$ and take $u_0 = A^{-1}f_0$.

(ii) Note that Re spec$(A) > 0$ and therefore $||e^{-At}|| \le Me^{-\omega t}$, $t \ge 0$, for some $M, \omega > 0$, where M, ω are independent of t.

Problem 2: (Positive spectrum of A doesn't imply exponential decay of e^{-At}). Let u be a measurable function on $[0,\infty[$ and set

$$||u||_1 = \int_0^\infty e^s |u(s)| ds.$$

Let E be the space of all measurable functions u on $[0,\infty[$ such that $||u||_1 < \infty$. Introduce the Banach space $X = E \cap L_2(0,\infty)$ with $||u|| = ||u||_1 + ||u||_{L_2}$. Define on X the semigroup $T(t)$ as follows:

$$T(t)u(x) = u(t+x), \qquad t \ge 0.$$

(i) Show that $T(t)$ is a C_0-semigroup (see problem 1, sheet 1) and $||T(t)|| = 1$ for $t \ge 0$.

(ii) Prove that the generator of $T(t)$ is $Au = u'$ and its domain is $D(A) = \{u \in X : u$ is absolutely continuous, $u' \in X\}$.

(iii) Show that the set $\{\lambda : Re\lambda > -1\}$ is a subset of $\rho(A)$.

Problem 3: Discuss how analytic semigroups can be applied to the equation

$$u_{tt} = \Delta u,$$

with Dirichlet boundary conditions.