Homework assignment

Differentialgleichungen III

Problem Sheet 6

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http://dynamics.mi.fu-berlin.de/lectures/13SS-Gurevich-Dynamics/

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Problem 1: Consider the semilinear Cauchy problem

$$\begin{cases} \dfrac{du}{dt} + Au = f(t,u), & t \in (0,t_1), \\ u(0) = u_0. \end{cases} \tag{1}$$

Assume that

(i) A is a sectorial operator, $f : U \subset \mathbb{R} \times X^\alpha \to X$ is locally Lipschitz in u and Hölder in t, where $\alpha \in [0,1)$ and $(0,u_0) \in U$.

(ii) f maps closed bounded subset of U to bounded subset of X.

(iii) $u(t)$ is a solution of (1) on $(0,t_1)$, where t_1 is *maximal, i.e.* $\forall t_2 > t_1$ *there is no solution of (1) on* $(0,t_2)$.

Show that either $t_1 = \infty$, or $\exists t_n \nearrow t_1$ such that $(t_n, u(t_n)) \to \partial U$ as $n \to \infty$ (If U is unbounded, then ∂U contains ∞.)

Hints:

(i) Let B be a closed subset of U. Assume that $t_1 < \infty$ and $u(t) \in B$ as $t \to t_1$. Take $\beta \in (\alpha, 1)$ and show that $||u(t)||_\beta \leq$ const as $t \to t_1$.

(ii) Show that $||u(t) - u(\tau)|| \leq \text{const}|t-\tau|^{\beta-\alpha}$ for all t, τ near t_1.

(iii) Deduce that $u(t) \to u_1 \in B$ in X^α as $t \to t_1$, and extend the solution with $u(t_1) = u_1$.

Problem 2: Consider problem (1). Assume that

(i) A is a sectorial operator. $f : U \subset \mathbb{R} \times X^\alpha \to X$ is locally Lipschitz in u and Hölder in t, where $\alpha \in [0, 1)$ and $(0, u_0) \in U$.

(ii) A has a compact resolvent.

(iii) If B is bounded and closed in X^α and $\mathbb{R}_+ \times B \subset U$, then $f(\mathbb{R}_+ \times B)$ is bounded in X.

(iv) $u(t)$ is a solution of (1) for $t \in (0, \infty)$.

(v) The set $\{u(t) : t \in [0, \infty)\}$ is bounded in X^α.

Show that the set $\{u(t) : t \in [0, \infty)\}$ is precompact in X^α.
Hint: Take $\beta > \alpha$ and show that $||u(t)||_\beta \leq$ const for all $t \geq 1$, and hence $\{u(t) : t \in [1, \infty)\}$ is precompact in X^α.

Problem 3: **Delay equations.** Let A be sectorial in a Banach space X, U an open set in $\mathbb{R} \times X^\alpha \times X^\alpha$, and $f : U \to X$ be a locally Lipschitz function. Consider the problem

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) = f(t, u(t), u(t-1)), & t > 0, \\ u(t) = \varphi(t), & t \in [-1, 0]. \end{cases}$$

Prove local existence and uniqueness when φ is Hölder continuous from $[-1, 0]$ to X^α.

Problem 4: **Coupled parabolic equation and ordinary equation.** Let X, Y be Banach spaces. Let A be sectorial in X. Let $f : U \subset \mathbb{R} \times X^\alpha \times Y$ be a locally Lipschitz function, where $\alpha \in [0, 1)$ and U is open. Prove the local existence and uniqueness results for

$$\begin{cases} \dfrac{du}{dt} + Au = f(t, u, v), \\ \dfrac{dv}{dt} = g(t, u, v). \end{cases}$$