Homework assignment

Differentialgleichungen III
Problem Sheet 7

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http://dynamics.mi.fu-berlin.de/lectures/13SS-Gurevich-Dynamics/

Tutorial discussion date: Friday, June 6, 2013, at 10:00am

Problem 1: **Definition:** Let $S(t),\ t \geq 0$, be a dynamical system on C. For any $u \in C$ we define $\gamma(u) = \{S(t)u, t \geq 0\}$ to be the *orbit* through u. We call u an *equilibrium* if $\gamma(u) = \{u\}$, and a *periodic orbit* if there exists $p > 0$ such that $\gamma(u) = \{S(t)u, 0 \leq t \leq p\} \neq \{u\}$.

An orbit $\gamma(u)$ (or sometimes the point u) is *stable* if $S(t)y \to S(t)u$ as $y \to u$, $y \in C$, uniformly in $t \geq 0$. An orbit $\gamma(u)$ is *unstable* if it is not stable. An orbit $\gamma(u)$ is *uniformly asymptotically stable* if it is stable and also there is a neighborhood $V = \{y \in C : dist(u, y) < r\}$ such that $dist(S(t)y, S(t)u) \to 0$ as $t \to \infty$, uniformly for $y \in V$.

Prove the following statements:

(i) If $\gamma(u)$ is unstable, then so is $\gamma(y)$ for any $y \in \gamma(u)$. Does the corresponding result hold when $\gamma(u)$ is stable?

(ii) If $y \in \gamma(u)$ and $\gamma(y)$ is stable, then $\gamma(u)$ is stable.

(iii) If $\gamma(u)$ is stable, then it is also *orbitally stable*, i.e. whenever $y \to u$, $y \in C$, we have $dist\{S(t)y, \gamma(u)\} \to 0$ uniformly in $t \geq 0$.

Problem 2: Let $\{S(t), t \geq 0\}$ be a dynamical system on C, and let B be open in C. Let $S_B(t)$ be the restriction of $S(t)$ to B. Then for each $u \in B$, there exists a maximal $T(u)$, $0 < T(u) \leq \infty$, such that $\{S_B(u), 0 \leq t \leq T(u)\}$ is in B. The pair $\{S_B, T\}$ is called a *local dynamical system*. Show that this family of maps satisfies

(i) If $u_n \to u_0$ in B and $0 \leq t < T(u_0)$, then there exists $N > 0$ such that $T(u_n) > t$ for $n \geq N$, and $S_B(t)u_n \to S_B(t)u_0$ in B.

(ii) If $u \in B,\ t \to S_B(t)u$ is continuous for $0 \leq t < T(u)$ into B.

(iii) $S_B(0) =$ identity on B.

(iv) If $u \in B$, $t, \tau \geq 0$ and $t + \tau < T(u)$ then $S_B(t)(S_B(\tau)u) = S_B(t + \tau)u$.

Problem 3:

(i) Let A be sectorial in X, and $f : X^\alpha \to X$ be locally Lipschitz for some $\alpha < 1$. Show that the equation

$$\frac{du}{dt} + Au = f(u),$$

defines a local dynamical system $\{S_B, T\}$ on any open set $B \subset X^\alpha$.

(ii) Consider the same A, f as in (i), and a local dynamical system $\{S_B, T\}$ for some open set $B \subset X^\alpha$. Assume in addition that $||f(u)|| \leq M < \infty$ for all $u \in B$. Define $g(u) = f(u)\phi(||f(u)||)$ for $u \in X^\alpha$, where $\phi(r) = 1$ if $0 \leq r \leq M$, $\phi(r) = M/r$ if $r \geq M$. Note that ϕ is Lipschitz continuous in $\mathbb{R}_+$. Show that

$$\frac{du}{dt} + Au = g(u)$$

defines a dynamical system on X^α whose restriction to B is $\{S_B, T\}$.